C C

Math 1554 Linear Algebra Spring 2024

Midterm 3 (C) Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: Key ____________________ GTID Number: ____________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) ________ TA Name ____________________

Circle your instructor:

Prof Barone Prof Belegradek Prof Kumar Prof Sun

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 9 pages of questions.

Midterm 3 (C) Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is a real $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	●	If A is an $n \times n$ real diagonalizable matrix, then A has n distinct eigenvalues.
		e.g. $A = I$
○	●	If $\text{proj}_{\vec{u}}(\vec{v}) = \vec{0}$, then $\{\vec{u}, \vec{v}\}$ is linearly independent.
		e.g. $\vec{v} = \vec{0}$
○	●	If $A = PDP^{-1}$ is diagonalizable and invertible, then A^{-1} is diagonalizable with the same matrix D.
		$A^{-1} = PD^{-1}P^{-1}$ D^{-1} not D.
●	○	If A has QR-factorization $A = QR$, then $\text{Nul}(A^T) = \text{Nul}(Q^T)$.
		$\text{Nul}\, A^T = (\text{Col}\, A)^\perp = (\text{Col}\, Q)^\perp = \text{Nul}\, Q^T$ ✓
●	○	If $A\vec{x} = \vec{x}$ and $A\vec{y} = 2\vec{y}$ then the matrix $B = [\vec{x}\ \vec{y}]$ has rank 2.
		$\{\vec{x}, \vec{y}\}$ linearly independent
○	●	Every 4×4 real matrix A has a real eigenvalue.
		$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ & $A = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$
●	○	If U is an $n \times n$ orthogonal matrix, then $UU^T = I_n$.
		$U^T = U^{-1}$
○	●	If A is $n \times n$ and A has n distinct real eigenvectors, then A is diagonalizable.

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \ldots$ are all eigenvectors w/ $\lambda = 1$

You do not need to justify your reasoning for questions on this page.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
●	○	A 2×2 matrix A such that A is an orthogonal matrix and $\det(A) = -1$.
●	○	A 3×3 matrix in RREF that has exactly one pivot and is not diagonalizable.
○	●	A real 5×5 matrix A whose eigenvalues are precisely $i, 1, 1+i, 1-i, 0$.
●	○	A real 5×5 matrix A with exactly two real eigenvalues, and the two eigenspaces of A have dimension 2 and dimension 3, respectively.

e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

e.g. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\bar{i} = -i$ must also be eigenvalue.

e.g. $A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) (2 points) Which of the following are examples of a matrix A which satisfy the following property. The matrix A is not diagonalizable while the matrix A^2 is diagonalizable. *Select all that apply.*

● $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \to A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ diag'ble ✓ but A has geo = 1, alg = 2 for $\lambda = 0$ so not diag'ble ✓

○ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \lambda_1 = 1, \lambda_2 = -1$ A diag'ble ✗

● $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to$ A not diag'ble, no real eigenvalues ✓

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ diag'ble ✓

○ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\to \lambda_1 = 2, \lambda_2 = 0$ A diag'ble ✗

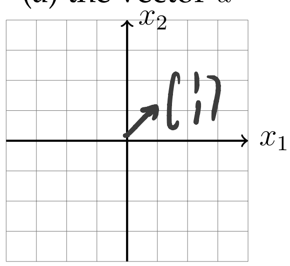
$$\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-2\\0\end{bmatrix}$$

Midterm 3 (C) Make-up. Your initials: ______________

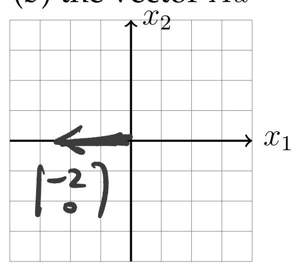
You do not need to justify your reasoning for questions on this page.

2. (5 points) Consider $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Sketch (a) the vector $\vec{u}$ and (b) the vector $A\vec{u}$. Then, fill in the blanks for the following statements to make the statements true.

(a) the vector $\vec{u}$



(b) the vector $A\vec{u}$



(c) The transformation defined by $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ is a rotation-dilation where the rotation is through $\boxed{135^\circ}$ (degrees or radians counter-clockwise) and the scaling factor is $\boxed{\sqrt{2}}$.

(d) An eigenvector associated to the eigenvalue $\lambda = -1+i$ of A is the vector $\vec{x} = \begin{pmatrix} \boxed{i} \\ 1 \end{pmatrix}$.

$$A-\lambda I = \begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix} - \begin{bmatrix}-1+i & 0\\ 0 & -1+i\end{bmatrix} = \begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix} \sim \begin{bmatrix}1 & -i\\ 0 & 0\end{bmatrix} \quad \vec{x}=s\begin{bmatrix}i\\1\end{bmatrix}$$

3. (3 points) Fill in the blanks so that the following statements are true. Let $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) An eigenvector of A corresponding to eigenvalue $\lambda = 4$ is $\vec{x} = \begin{pmatrix} \boxed{6} \\ 3 \\ \boxed{3} \end{pmatrix}$.

(b) The vector $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to eigenvalue $\lambda = \boxed{3}$.

(c) The characteristic polynomial of A is equal to $\boxed{p(\lambda) = -(\lambda-3)^2(\lambda-4)}$.

You may leave your answer in factored form.

Math 1554 Linear Algebra, Midterm 3 (C) Make-up. Your initials: ______________
You do not need to justify your reasoning for questions on this page.

4. (6 points) Fill in the blanks.

   (a) Suppose $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ and that $\vec{u}\cdot\vec{v} = -4$, $\|\vec{u}\| = \sqrt{6}$ and $\|\vec{v}\| = 2\sqrt{2}$. Determine the length of $\vec{2u} - 3\vec{v}$.

$$\|2\vec{u} - 3\vec{v}\| = \boxed{12}$$

$(2u-3v)\cdot(2u-3v)$

$= 4u\cdot u - 12u\cdot v + 9v\cdot v$

$= 4(\sqrt{6})^2 - 12(-4) + 9(2\sqrt{2})^2$

$= 4\cdot 6 + 48 + 9\cdot 8 = 24+48+72 = 144$

So $\|2u-3v\| = \sqrt{(2u-3v)\cdot(2u-3v)}$

$= \sqrt{144} = 12$

   (b) A 3×2 matrix such that $\mathrm{Col}(A)^\perp$ is spanned by $\begin{pmatrix}1\\0\\1\end{pmatrix}$.

$$A = \begin{pmatrix}0 & 1\\ 1 & 0\\ 0 & -1\end{pmatrix}$$

needs to have 2 pivots and all columns orthogonal to $\begin{bmatrix}1\\0\\1\end{bmatrix}$.

   (c) Write $\vec{y} = \vec{x} + \vec{z}$ as the sum of two vectors, where $\vec{x}$ in span$\{\vec{u}\}$ and $\vec{z}$ is orthogonal to $\vec{u}$.

$$\vec{y} = \begin{pmatrix}5\\3\end{pmatrix} \qquad \vec{u} = \begin{pmatrix}-1\\1\end{pmatrix} \qquad \vec{x} = \begin{pmatrix}1\\-1\end{pmatrix} \qquad \vec{z} = \begin{pmatrix}4\\4\end{pmatrix}$$

$\vec{x} = \mathrm{proj}_u(y) = \frac{y\cdot u}{u\cdot u}u = \frac{-5+3}{2}\begin{bmatrix}-1\\1\end{bmatrix} = -\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

$\vec{z} = \vec{y} - \vec{x} = \begin{bmatrix}5\\3\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix}$

You do not need to justify your reasoning for questions on this page.

5. (4 points) Let W be the subspace in $\mathbb{R}^4$ which is the set of solutions to the following homogeneous linear equation. Find $\dim W$ and an orthonormal basis for $W^\perp$.

$$x_1 - x_2 + x_3 - 3x_4 = 0$$

$W = \text{Nul } A$ where

$A = \begin{bmatrix} 1 & -1 & 1 & -3 \end{bmatrix}$

$\dim W = \boxed{3}$

orthonormal basis for $W^\perp$ is $\boxed{\left\{ \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix} \right\}}$

$W^\perp = (\text{Nul } A)^\perp = \text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix} \right\}.$

So an orthonormal basis for $W^\perp$ is

$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{1+1+1+9}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix}$$

$\sqrt{12} = 2\sqrt{3}$

Since $W = \text{Nul } A$ and rank $(A) = 1$ & # free vars is 3,

$\dim W = 3.$

Midterm 3 (C) Make-up. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

6. (4 points) The matrix A has the QR factorization $A = QR$. Find the least-squares solution of the linear system $A\vec{x} = \vec{b}$ when

$$Q = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 0 & 0 \\ -12 & 5 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}.$$

$A = QR$ and $Q^TQ = I$.

$\hat{x} = \boxed{\begin{bmatrix} 13/3 \\ 5/3 \end{bmatrix}}$

Normal equations

$A^TA\hat{x} = A^T\vec{b}$ becomes

$(QR)^TQR\hat{x} = (QR)^T\vec{b}$

$\Rightarrow \cancel{R^T}\cancel{Q^T}QR\hat{x} = \cancel{R^T}Q^Tb$

$\Rightarrow \boxed{R\hat{x} = Q^Tb}$ ← Solve this.

$Q^Tb = \frac{1}{13}\begin{bmatrix} 5 & 0 & -12 \\ 12 & 0 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 13 \\ 65 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 5/3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 13/3 \\ 0 & 1 & 5/3 \end{array}\right]$$

Ans.

$$\boxed{\hat{x} = \begin{bmatrix} 13/3 \\ 5/3 \end{bmatrix}}$$

Midterm 3 (C) Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

7. (4 points) Set $A = \begin{pmatrix} 0 & 0 \\ 2 & 3 \\ 4 & 6 \end{pmatrix}$. Find a basis for each of the four fundamental subspaces $\mathrm{Col}(A)$, $\mathrm{Nul}(A)$, $\mathrm{Row}(A)$ and $\mathrm{Nul}(A^T)$.

① $\mathrm{Col}\,A = \mathrm{Span}\left\{\begin{pmatrix}0\\2\\4\end{pmatrix}, \begin{bmatrix}0\\3\\6\end{bmatrix}\right\} = \mathrm{Span}\left\{\begin{bmatrix}0\\2\\4\end{bmatrix}\right\}$

basis for $\mathrm{Col}(A)$ is $\left\{\begin{bmatrix}0\\2\\4\end{bmatrix}\right\}$

② $\mathrm{Nul}\,A \quad A \sim \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \vec{x} = s\begin{pmatrix}-3/2\\1\end{pmatrix}$

s

basis for $\mathrm{Nul}(A)$ is $\left\{\begin{bmatrix}-3/2\\1\end{bmatrix}\right\}$

③ $\mathrm{Row}\,A = \mathrm{span}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix}\right\} = \mathrm{Span}\left\{\begin{bmatrix}2\\3\end{bmatrix}\right\}$

basis for $\mathrm{Row}(A)$ is $\left\{\begin{bmatrix}2\\3\end{bmatrix}\right\}$

④ $\mathrm{Nul}\,A^T = (\mathrm{Col}\,A)^\perp$

need two vectors which are linearly independent & orthogonal to $\begin{bmatrix}0\\2\\4\end{bmatrix}$

e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}$ & $\begin{bmatrix}0\\-4\\2\end{bmatrix}$

basis for $\mathrm{Nul}(A^T)$ is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{bmatrix}0\\-4\\2\end{bmatrix}\right\}$

Check $V_1 \cdot V_2 = 0$?

$1-3+1+1=0$ ✓

Midterm 3 (C) Make-up. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

8. (5 points) **Show all work for problems on this page.**

   Find the vector $\vec{x}$ which is in W and closest to $\vec{y}$, where the subspace W is spanned by $\vec{v}_1$ and $\vec{v}_2$. Then, compute the distance from $\vec{y}$ to W.

$$\vec{y} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\hat{x} = \mathrm{proj}_{\{\vec{v}_1, \vec{v}_2\}}(\vec{y})$$

$$= \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{3+9+3}{1+9+1+1} \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix} + \frac{9-3+3}{9+1+1+1} \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{15}{12} \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix} + \frac{9}{12} \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{4} \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14/4 \\ 12/4 \\ -2/4 \\ 8/4 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3 \\ -1/2 \\ 2 \end{bmatrix}$$

$$\vec{x} = \boxed{\begin{bmatrix} 7/2 \\ 3 \\ -1/2 \\ 2 \end{bmatrix}}$$

$$\mathrm{dist}(\vec{y}, W) = \boxed{\sqrt{3/2}}$$

$$\mathrm{dist}(y, W) = \| \vec{y} - \vec{x} \| = \left\| \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 7/2 \\ 3 \\ -1/2 \\ 2 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{3/2}$$

9. (5 points) **Show all work for problems on this page.**

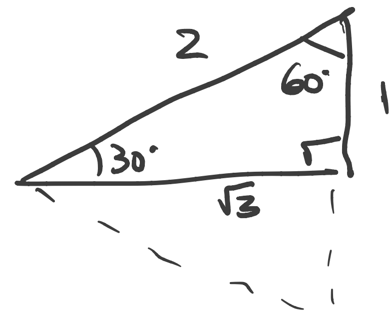
Compute the parameters α and β of the least-squares curve $y = \alpha x + \beta \cos(\frac{\pi x}{3})$ that best fits the data below.

x	-2	0	1
y	-1	0	2

$\alpha = \boxed{5/7}$ $\beta = \boxed{2/7}$

model is

$$y = \alpha x + \beta \cos\left(\frac{\pi x}{3}\right)$$



Plug in data.

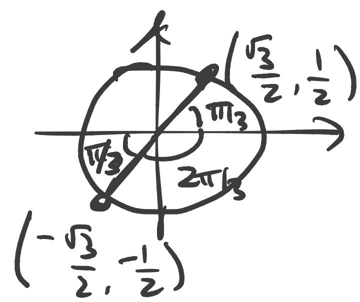
@ $(-2,-1)$ $\quad -2\alpha + \beta\cos\left(\frac{-2\pi}{3}\right) = -1$

@ $(0,0)$ $\quad 0\alpha + \beta\cos(0) = 0$

@ $(1,2)$ $\quad \alpha + \beta\cos(\pi/3) = 2$

$\cos(\pi/3) = 1/2$

$\cos\left(\frac{-2\pi}{3}\right) = -1/2$



$$A = \begin{bmatrix} -2 & -1/2 \\ 0 & 1 \\ 1 & 1/2 \end{bmatrix} \qquad A^TA = \begin{bmatrix} -2 & 0 & 1 \\ -1/2 & 1 & 1/2 \end{bmatrix}\begin{pmatrix} -2 & -1/2 \\ 0 & 1 \\ 1 & 1/2 \end{pmatrix}$$

$$b = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \qquad = \begin{pmatrix} 5 & 3/2 \\ 3/2 & 3/2 \end{pmatrix}$$

Next.

Solve $A^TA\hat{x} = A^Tb$, so reduce

$$A^Tb = \begin{pmatrix} -2 & 0 & 1 \\ -1/2 & 0 & 1/2 \end{pmatrix}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{pmatrix} 4 \\ 3/2 \end{pmatrix} \qquad \hat{x} = \begin{pmatrix} 5/7 \\ 2/7 \end{pmatrix}$$

$$[A^TA \mid A^Tb] = \left[\begin{array}{cc|c} 5 & 3/2 & 4 \\ 3/2 & 3/2 & 3/2 \end{array}\right] \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 5 & 3/2 & 4 \end{array}\right) \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -7/2 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2/7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5/7 \\ 0 & 1 & 2/7 \end{array}\right]$$

This page may be used for scratch work. Please indicate clearly on the problem if you would like your work on this page to be graded. Loose scrap paper is not permitted. This page must **NOT** ***be detached*** *from your exam booklet at any time.*

This page may be used for scratch work. Please indicate clearly on the problem if you would like your work on this page to be graded. Loose scrap paper is not permitted. This page must **NOT** ***be detached*** *from your exam booklet at any time.*