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Math 1554 Linear Algebra Spring 2024

Midterm 2 (C) Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: Key ______________ GTID Number: ______________

Student GT Email Address: ______________@gatech.edu

Section Number (e.g. A3, G2, etc.) ________ TA Name ______________

Circle your instructor:

Prof Barone Prof Belegradek Prof Kumar Prof Sun

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

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You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is a real $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	●	If A is $n \times n$ and has LU-factorization $A = LU$, then U is invertible.
○	●	If $AB\vec{x} = \vec{0}$ has a nontrivial solution and B is not the zero matrix, then A is not invertible.
●	○	If T is a one-to-one linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ and $\mathcal{H}$ is the set of vectors $\vec{b}$ such that $T(\vec{x}) = \vec{b}$ for some $\vec{x} \in \mathbb{R}^4$, then $\dim \mathcal{H} = 4$.
○	●	If $\vec{v}$ is an eigenvector of A with eigenvalue $\lambda = 4$, then $2\vec{v}$ is an eigenvector of A with eigenvalue $\lambda = 8$.
●	○	If $\vec{v}$ and $\vec{w}$ are eigenvectors of a square matrix A with the same eigenvalue and $\vec{v} \neq -\vec{w}$, then $\vec{v} + \vec{w}$ is also an eigenvector of A.
●	○	If A and B are $n \times n$ matrices with $\det(AB) \neq 0$, then A and B are row equivalent.
○	●	If $\vec{x}$ and $\vec{y}$ are probability vectors, then $\vec{x} + \vec{y}$ is a probability vector.
●	○	If A and B are $n \times n$ and $\det(AB) = \frac{1}{2}$, then A is invertible.

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(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
●	○	A probability vector $\vec{x}_0$ and a stochastic matrix P such that the Markov chain $\vec{x}_k = P^k\vec{x}_0$, $k \geq 0$, has no limit as k tends to infinity.
●	○	An 3×6 matrix A such that the span of the first two columns of A is 2-dimensional, and the span of the last two columns of A is also 2-dimensional.
●	○	A square matrix A such that $\det(A) = 0$ and A none of the rows of A are scalar multiples.
○	●	An invertible 3×3 matrix A such that $\det(A + A) = 2\det(A)$.

(c) (2 points) The column space of a matrix A is spanned by the following two vectors

$$\vec{u} = \begin{pmatrix} 1 \\ -3 \\ 4 \\ -3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ -6 \\ 8 \\ -6 \end{pmatrix}$$

and the null space of A has dimension 5. Which one of the following is **true**? *Choose only one.*

● The system $A\vec{x} = \vec{v}$ is consistent.

○ The first two columns of A are linearly independent.

○ The null space of A is a subspace of $\mathbb{R}^5$.

○ The range of $T(\vec{x}) = A\vec{x}$ is a plane in $\mathbb{R}^4$.

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2. (2 points) Give a matrix A whose null space is spanned by the vector $\binom{2}{3}$ and whose column space is spanned by $\begin{pmatrix}1\\1\\3\end{pmatrix}$. If this is not possible, write NP in the box.

$$A = \boxed{\begin{pmatrix} 3 & -2 \\ 3 & -2 \\ 9 & -6 \end{pmatrix}}$$

3. (3 points) Suppose A, B and C are invertible $n \times n$ matrices. Find the inverse of the partitioned matrix

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{pmatrix}.$$

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}$$

① $AX + BZ = I$ $\xrightarrow{③}$ $BZ = I \Rightarrow Z = B^{-1}$

② $AY + BW = 0$ $\xrightarrow{④}$ $AC^{-1} + BW = 0 \Rightarrow AC^{-1} = -BW \Rightarrow -B^{-1}AC^{-1} = W$

③ $CX = 0 \Rightarrow X = 0$ (since C invertible)

④ $CY = I_n \Rightarrow Y = C^{-1}$

4. (3 points) Let $\mathcal{S}$ be a triangle in $\mathbb{R}^2$ with area 4. Find the standard matrix A of

$$T(x_1, x_2) = (x_1 + 4x_2, x_1 + 2x_2),$$

as well as the area of the image of $\mathcal{S}$ under the transformation T, and the area of the image of $\mathcal{S}$ under T^{-1}.

$A = \boxed{\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}}$ area$(T(\mathcal{S})) = \boxed{8}$ area$(T^{-1}(\mathcal{S})) = \boxed{2}$

$\det A = 2 - 4 = -2$

$= |\det A| * \text{area}(S)$

$= 2 * 4$

$= |\det A^{-1}| * \text{area}(S)$

$= \frac{1}{2} * 4$

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5. (2 points) Solve $A\vec{x} = \vec{b}$ if A has LU-factorization $A = LU$ and

$$U = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L^{-1}\vec{b} = \begin{pmatrix} 4 \\ -6 \\ 4 \\ 0 \end{pmatrix}.$$

$LU\vec{x} = \vec{b}$

$\Rightarrow U\vec{x} = L^{-1}\vec{b}$

$$\vec{x} = \boxed{\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

6. (3 points) Find all eigenvalues of the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 2 & -1 \\ -1 & -4 & 2 \end{pmatrix}$.

$\lambda_1 = \boxed{0}$ $\qquad \lambda_2 = \boxed{2}$ $\qquad \lambda_3 = \boxed{4}$

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ -2 & 2-\lambda & -1 \\ -1 & -4 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{vmatrix} = -(\lambda-2)\left((\lambda-2)^2 - 4\right)$$

$$= -(\lambda-2)(\lambda^2 - 4\lambda) = -(\lambda-2)(\lambda-4)\lambda$$

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You do not need to justify your reasoning for questions on this page.

7. (2 points) If possible, fill in the box with the missing element of the vectors $\vec{v}$ and $\vec{x}$ with a number so that $\vec{v}$ belongs to span $\{\vec{u}, \vec{w}\}$ and $\vec{x}$ belongs to Nul(A). If it is not possible write `NP` in the space.

$3\vec{u} - \vec{w} = \vec{v}$ ✓

rank $A = 2$

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} \boxed{1} \\ 3 \\ 2 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} \boxed{NP} \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}. \Rightarrow \text{Nul} A = \{\vec{0}\}$$

$$\left[\vec{u}\ \vec{w} \,\middle|\, \vec{v}\right] = \left[\begin{array}{cc|c} 1 & 2 & h \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{array}\right]$$

$-h + 1 = 0 \Rightarrow h = 1$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & h \\ 1 & 0 & 3 \\ 0 & -1 & 2-h \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 2 & h-3 \\ 1 & 0 & 3 \\ 0 & 1 & h-2 \end{array}\right] \sim \left(\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & h-2 \\ 0 & 0 & h-3-2(h-2) \end{array}\right]\right)$$

8. (3 points) Find an eigenvector $\vec{v}$ of A corresponding to the eigenvalue $\lambda = 5$.
Hint: check your answer.

$$A = \begin{pmatrix} 6 & -3 \\ 3 & -4 \end{pmatrix}$$

$$A - 5I = \begin{bmatrix} 6-5 & -3 \\ 3 & -4-5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 3 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\vec{v} = \boxed{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

9. (6 points) **Show all work for problems on this page.**

Suppose $A \in \mathbb{R}^{3\times 3}$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$ and $\lambda_3 = 0$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors corresponding to λ_1, λ_2, and λ_3, respectively:

$$\vec{v}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Write the answers to each of the following parts in the designated boxes given below.

(a) What is the dimension of the null space of $A + 2I_3$?
(b) If $\vec{w} = \vec{v}_1 + 2\vec{v}_2 - 5\vec{v}_3$, what is $A\vec{w}$? *You may leave your answer in terms of* $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
(c) What is $\det(A)$?

(b)
$$A\vec{w} = A(\vec{v}_1 + 2\vec{v}_2 - 5\vec{v}_3)$$
$$= A\vec{v}_1 + 2A\vec{v}_2 - 5A\vec{v}_3$$
$$= \vec{v}_1 - 4\vec{v}_2 - 0\vec{v}_3$$

(a) $\dim \mathbf{Nul}(A + 2I_3) = \boxed{1}$

(b) $A\vec{w} = \boxed{\vec{v}_1 - 4\vec{v}_2}$

(c) $\det(A) = \boxed{0}$

(c) $\det A = 0$ b/c $\lambda = 0$ is an eigenvalue of A (IMT).

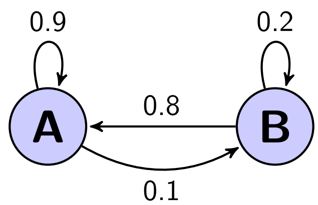
10. (6 points) **Show your work for part (b) on this page.**

Use the following Markov chain diagram to answer the questions.

(a) Find the stochastic matrix P of the Markov chain.

(b) Find the unique steady state probability vector $\vec{q}$ of P.

(c) If $\vec{x}_0 = \begin{pmatrix} .8 \\ .2 \end{pmatrix}$, find the limit $\vec{l}$ of the sequence of vectors $P^k\vec{x}_0$ as k tends to infinity.



(a) $P = \begin{bmatrix} .9 & .8 \\ .1 & .2 \end{bmatrix}$

$P = \boxed{\begin{bmatrix} .9 & .8 \\ .1 & .2 \end{bmatrix}}$

(b) P is regular so $P\vec{q} = \vec{q}$ for a unique prob vector $\vec{q} \in \text{Nul}(P - I)$.

$P - I = \begin{bmatrix} -.1 & .8 \\ .1 & -.8 \end{bmatrix} \sim \begin{bmatrix} 1 & -8 \\ 0 & 0 \end{bmatrix}$ $\quad \vec{x} = s\begin{pmatrix} 8 \\ 1 \end{pmatrix}$

$\vec{q} = \begin{pmatrix} 8/9 \\ 1/9 \end{pmatrix}$

$\vec{q} = \boxed{\begin{pmatrix} 8/9 \\ 1/9 \end{pmatrix}}$

(c) Since P is regular, for any $\vec{x}_0$ prob vector,

$\lim_{k \to \infty} P^k \vec{x}_0 = \vec{q}$.

$\vec{l} = \boxed{\begin{bmatrix} 8/9 \\ 1/9 \end{bmatrix}}$.

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11. (6 points) **Show all work for problems on this page.**

Find a basis for the subspace H of $\mathbb{R}^4$. Clearly show all your steps for credit.
Put your answer in the box.

$$H = \{\vec{x} \in \mathbb{R}^4 \mid 2x_1 + 4x_2 + 4x_3 - 8x_4 = 0\}$$

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

$$A = \begin{bmatrix} 2 & 4 & 4 & -8 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} & r & s & t \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & -4 \end{bmatrix} \end{array}$$

$x_1 = -2x_2 - 2x_3 + 4x_4$

$x_2 = r$ (free)

$x_3 = s$ (free)

$x_4 = t$ (free)

$$\vec{x} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This page may be used for scratch work. Please indicate clearly on the problem if you would like your work on this page to be graded. Loose scrap paper is not permitted. This page must **NOT** ***be detached*** *from your exam booklet at any time.*