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Math 1554 Linear Algebra Spring 2024

Midterm 1 (C) Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: Key ______________________ GTID Number: ____________________

Student GT Email Address: ______________________@gatech.edu

Section Number (e.g. A3, G2, etc.) ________ TA Name ____________________

Circle your instructor:

Prof Barone Prof Belegradek Prof Kumar Prof Sun

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	●	If the last row of the RREF of a matrix A is all zeros, then $A\vec{x} = \vec{0}$ has infinitely many solutions.
●	○	If $\vec{x} \neq \vec{0}$ and $T(\vec{x}) = \vec{0}$, then T is not one-to-one.
○	●	If the linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one, then $A\vec{x} = \vec{b}$ has a unique solution.
●	○	If A is a 2×2 matrix that represents a clockwise rotation by 180 degrees and I is the 2×2 identity matrix, then $A^2 = I$.
○	●	If the linear system $A\vec{x} = \vec{b}$ is inconsistent, then its augmented matrix $[A \mid \vec{b}]$ has a pivot in each column in its echelon form.
●	○	The set of solutions to the linear system below is a line in $\mathbb{R}^4$. $\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 0 \\ -x_1 + 3x_2 - x_3 + x_4 = 0 \end{cases}$
○	●	If $\vec{v}$ is a solution to an inhomogeneous system $A\vec{x} = \vec{b}$, then the vector $\vec{v} + \vec{v}$ is also a solution to $A\vec{x} = \vec{b}$.
○	●	If A, B are $n \times n$ matrices, then $B(A + I)B = B^2A + B^2$.

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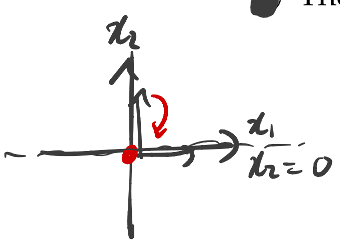
You do not need to justify your reasoning for questions on this page.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
●	○	The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ is one-to-one, but T is not onto.
●	○	A transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that is one-to-one.
●	○	A homogeneous linear system which has solutions that are not the trivial solution.
●	○	Nonzero vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ such that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent and none of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are scalar multiples.

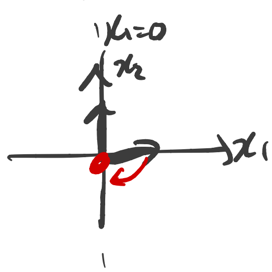
(c) (2 points) Let T and S be linear transformations of $\mathbb{R}^2$ such that T is the projection onto the line $x_2 = 0$ and S is the projection onto the line $x_1 = 0$. Which of following accurately describes the transformation $S(T(\vec{x}))$? Select only one.

○ Counterclockwise rotation by 180 degrees about the origin.
○ Projection onto the line $x_1 = 0$.
○ Projection onto the line $x_2 = 0$.
● The transformation that sends every vector to zero.



$T(\vec{e}_1) = \vec{e}_1$
$T(\vec{e}_2) = \vec{0}$

$S(T(\vec{e}_1)) = S(\vec{e}_1) = \vec{0}$
$S(T(\vec{e}_2)) = S(\vec{0}) = \vec{0}$



$S(\vec{e}_1) = \vec{0}$
$S(\vec{e}_2) = \vec{e}_2$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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You do not need to justify your reasoning for questions on this page.

(d) (4 points) For the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Which of the following sets are linearly independent? *Select all that apply.*

- ○ $\{\vec{v}_1, \vec{v}_2\}$
- ● $\{\vec{v}_2, \vec{v}_3\}$
- ● $\{\vec{v}_1, \vec{v}_3\}$
- ○ $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$

$2\vec{V}_1 = \vec{V}_2$

$V_1 + V_2 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} = 3\vec{U}_1$

2. (3 points) If possible, fill in the box with the missing element of the vector $\vec{x}$ so that $\vec{x}$ is in the set span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. If it is not possible write `NP` in the space.

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} -2 \\ -4 \\ \boxed{2} \end{pmatrix}$$

a

$$[A|\vec{x}] \quad \left[\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 2 & -1 & 5 & -4 \\ -1 & -2 & 0 & a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -7 & 7 & 0 \\ 0 & 1 & -1 & -2+a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2+a \end{array}\right]$$

$a = 2$

else $A\vec{y} = \vec{x}$

is inconsistent

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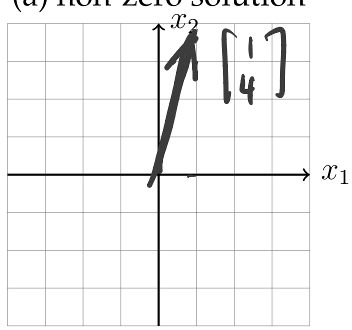
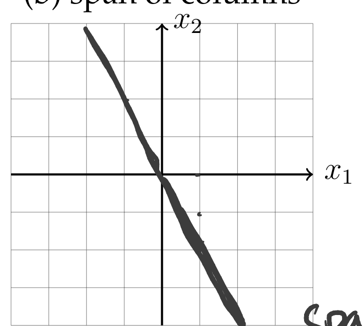
You do not need to justify your reasoning for questions on this page.

3. (4 points) Suppose $A = \begin{pmatrix} -4 & 1 \\ 8 & -2 \end{pmatrix}$ and sketch (a) a non-zero solution with integer entries to $A\vec{x} = \vec{0}$, and (b) the span of the columns of A.

(a) non-zero solution



(b) span of columns



span$\left\{\begin{bmatrix}1\\-2\end{bmatrix}\right\}$

$$\begin{bmatrix}-4 & 1\\ 8 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -1/4\\ 0 & 0\end{bmatrix} \quad \vec{x} = r\begin{pmatrix}1/4\\1\end{pmatrix} \quad (r=4) \quad \vec{x} = \begin{pmatrix}1\\4\end{pmatrix}$$

4. (4 points) Consider the linear system in variables x_1, x_2, x_3 with unknown constants below.

$$a_1x_1 + a_2x_2 + a_3x_3 = b_1$$
$$a_1x_1 + a_2x_2 + a_3x_3 = b_2$$

Which of the following statements about the solution set of this system are possible?
Select all that apply.

- ● The solution set is empty. $b_1 \neq b_2$
- ○ The solution set is a single point. two free vars
- ○ The solution set is a line. "
- ● The solution set is a plane. $b_1 = b_2$

5. (8 points) **Show your work for part (c) and put your answer in the boxes.**
Let T be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 - 3x_3 \\ x_1 - 2x_3 \\ x_1 + x_2 - x_3 \end{bmatrix}.$$

(a) What are the domain and codomain of T?

domain is $\boxed{\mathbb{R}^3}$

codomain is $\boxed{\mathbb{R}^3}$

(b) Find the matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

$$A = \boxed{\begin{bmatrix} 1 & -1 & -3 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix}}$$

(c) Find a vector $\vec{x}$ such that $T(\vec{x}) = \vec{b}$, where $\vec{b} = \left[\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right]$.

$$\vec{x} = \boxed{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -3 & 2 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 2 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

r

$$\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (r=0) \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

(d) Is T one-to-one? ○ yes ● no

(e) Is T onto? ○ yes ● no

6. (5 points) **Show all work for problems on this page.**

   For what value(s) of h is the following set of vectors linearly dependent?

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} h \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ h^2 \end{pmatrix} \right\}$$

$$h = \boxed{4, -1}$$

$$A = \begin{bmatrix} 1 & h & 2 \\ 0 & 2 & -3 \\ 2 & 0 & h^2 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 2 & -3 \\ 0 & -2h & h^2-4 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 2 & -3 \\ 0 & 0 & h^2-3h-4 \end{bmatrix}$$

Lin dependent cols if this is zero

otherwise 3 pivots in matrix A $\Rightarrow$ cols lin independent.

$$h^2 - 3h - 4 = 0$$

$$\Rightarrow (h-4)(h+1) = 0$$

$$\Rightarrow h = 4, -1$$

7. **Show your work in the space below the first box and put your answers in the boxes.**

   (a) (6 points) Write the parametric vector form for the general solution to the inhomogeneous equation $A\vec{x} = \vec{b}$ given below.

$$\begin{cases} 5x_1 + x_2 - 7x_3 - 9x_4 = 2 \\ 5x_1 + x_2 - 4x_3 - 6x_4 = -1 \end{cases}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1/5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2/5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 5 & 1 & -7 & -9 & 2 \\ 5 & 1 & -4 & -6 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 5 & 1 & -7 & -9 & 2 \\ 0 & 0 & 3 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 5 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$$\begin{array}{ccccc} & \swarrow r & & \swarrow s & \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1/5 & 0 & -2/5 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right] \qquad \vec{x} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1/5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2/5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

   (b) For the homogeneous system with the same coefficient matrix A as part (a) above, write down the parametric vector form of the general solution to $A\vec{x} = \vec{0}$.
   Hint: use your answer from part (a).

$$\vec{x} = r\begin{bmatrix} -1/5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2/5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

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