Reading Day Problem Bank
Chapters 1-4

Differentiation Rules to Remember!

- **Derivative of a Constant:** $\frac{d}{dx}(c) = 0$
- **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Constant Multiple Rule:** If c is a constant, then $\frac{d}{dx}[cf(x)] = cf'(x)$
- **Sum Rule:** $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
- **Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$
- **Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)g(x)}$
- **Derivative of the Natural Exponential Function:** $\frac{d}{dx}(e^x) = e^x$

Derivatives of the six basic trigonometric functions.

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\cos x) = -\sin x$ $\quad$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\quad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

The Chain Rule

If $f(u)$ is differentiable at u, and $u = g(x)$ is differentiable at x, then the derivative of the composite function $f \circ g(x) = f(g(x))$, with respect to x, is

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{OR} \quad \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Derivatives of Log and Exponential Functions If $b > 0$, then

- $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$
- $\frac{d}{dx}(b^x) = b^x \ln b$.

Derivatives of the inverse trigonometric functions.

$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ $\quad$ $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$ $\quad$ $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

1. Evaluate the following derivatives; you need not simplify your answers.

(a) $\frac{d}{dx}\left[x^2 \log_3(2x+1)\right]$

Solution. We can use the change of base formula (see **??**) to rewrite $\log_3(2x+1) = \frac{\ln(2x+1)}{\ln 3}$, so

$$\begin{aligned}
\frac{d}{dx}\left[x^2 \log_3(2x+1)\right] &= \frac{1}{\ln 3}\frac{d}{dx}\left(\boxed{x^2}\cdot\boxed{\ln(2x+1)}\right) \\
&= \frac{1}{\ln 3}\left(x^2\cdot\frac{d}{dx}\left(\boxed{\ln\left(\boxed{2x+1}\right)}\right) + 2x\ln(2x+1)\right) \\
&= \boxed{\frac{1}{\ln 3}\left(x^2\cdot\frac{1}{2x+1}\cdot 2 + 2x\ln(2x+1)\right)}
\end{aligned}$$

(b) *If* $y = (\tan 2)e^{\sqrt{\cos t}}$, *find* $\frac{dy}{dt}$.

Solution. Since $\tan 2$ is a constant,

$$\begin{aligned}
\frac{dy}{dt} &= (\tan 2)\cdot\frac{d}{dt}\left(\boxed{e^{\boxed{\sqrt{\cos t}}}}\right) \\
&= (\tan 2)e^{\sqrt{\cos t}}\cdot\frac{d}{dt}\left(\boxed{\boxed{(\cos t)}^{1/2}}\right) \\
&= (\tan 2)e^{\sqrt{\cos t}}\cdot\frac{1}{2}(\cos t)^{-1/2}(-\sin t)
\end{aligned}$$

(c) *If* $g(x) = \sec(x^2e^x)$, *find* $g'(x)$.

Solution.

$$\begin{aligned}
\frac{d}{dx}\left(\boxed{\sec\left(\boxed{x^2e^x}\right)}\right) &= \sec\left(x^2e^x\right)\tan\left(x^2e^x\right)\cdot\frac{d}{dx}\left(\boxed{x^2}\cdot\boxed{e^x}\right) \\
&= \sec\left(x^2e^x\right)\tan\left(x^2e^x\right)\left(x^2e^x + e^x\cdot 2x\right)
\end{aligned}$$

(Remember we use green boxes to indicate that we're using the Product Rule.)

(d) *If* $f(t) = \dfrac{t}{5\sqrt[3]{1+e^t}}$, *find* $f'(t)$. *(Can you rewrite* f *so that you don't need to use the Quotient Rule?)*

Solution. We can rewrite $f(t) = \frac{1}{5}t(1+e^t)^{-1/3}$, so

$$\begin{aligned}
f'(t) &= \frac{1}{5}\cdot\frac{d}{dt}\left(\boxed{t}\cdot\boxed{\left(1+e^t\right)^{-\frac{1}{3}}}\right) \\
&= \frac{1}{5}\left(t\cdot\frac{d}{dt}\left(\boxed{\boxed{\left(1+e^t\right)}^{-\frac{1}{3}}}\right) + \left(1+e^t\right)^{-\frac{1}{3}}\right) \\
&= \frac{1}{5}\left(t\left(-\frac{1}{3}\left(1+e^t\right)^{-\frac{4}{3}}\cdot\frac{d}{dt}\left(1+e^t\right)\right) + \left(1+e^t\right)^{-\frac{1}{3}}\right) \\
&= \frac{1}{5}\left(-\frac{1}{3}t\left(1+e^t\right)^{-\frac{4}{3}}e^t + \left(1+e^t\right)^{-\frac{1}{3}}\right)
\end{aligned}$$

(e) $f(x) = x\sin\left(x^3\right)$, $g(x) = x\sin^3 x$, *and* $h(x) = x(\sin x)^3$. *(Are these the same or different?)*

Solution. First, $h(x)$ is the same as $g(x)$. To differentiate $f(x)$ and $g(x)$, we start with the Product Rule:

$$\begin{aligned} f'(x) &= \sin\left(x^3\right) + x \cdot \frac{d}{dx}\left[\boxed{\sin\left(\boxed{x^3}\right)}\right] \\ &= \sin\left(x^3\right) + x \cdot \cos\left(x^3\right) \cdot 3x^2 \end{aligned}$$

and

$$\begin{aligned} g'(x) = h'(x) &= (\sin x)^3 + x \cdot \frac{d}{dx}\left[\boxed{\left(\boxed{\sin x}\right)^3}\right] \\ &= (\sin x)^3 + x \cdot 3(\sin x)^2 \cos x \end{aligned}$$

2. Compute derivatives for the following functions:

(a) $f(x) = \ln(\sec(\ln x))$.

Solution. We use the chain rule:

$$\begin{aligned} \frac{d}{dx}\left(\boxed{\ln\left(\boxed{\sec(\ln x)}\right)}\right) &= \frac{1}{\sec(\ln x)} \cdot \frac{d}{dx}\left(\boxed{\sec\left(\boxed{\ln x}\right)}\right) \\ &= \frac{1}{\sec(\ln x)} \cdot \sec(\ln x)\tan(\ln x) \cdot \frac{1}{x} \end{aligned}$$

(b) $f(x) = \arccos(e^{3-x^2})$.

Solution. We need the chain rule:

$$\begin{aligned} \frac{d}{dx}\left(\boxed{\arccos\left(\boxed{e^{3-x^2}}\right)}\right) &= -\frac{1}{\sqrt{1-(e^{3-x^2})^2}} \cdot \frac{d}{dx}\left(\boxed{e^{\boxed{3-x^2}}}\right) \\ &= -\frac{1}{\sqrt{1-(e^{3-x^2})^2}} \cdot e^{3-x^2} \cdot (-2x) \end{aligned}$$

(c) $f(x) = (\ln x)^{\ln x}$.

Solution. This is of the form $f(x)^{g(x)}$. So this would need logarithmic differentiation.

$$f(x) = (\ln x)^{\ln x}$$

Take natural log on both sides:

$$\begin{aligned} \ln[f(x)] &= \ln\left[(\ln x)^{\ln x}\right] \\ \ln[f(x)] &= \ln x \cdot \ln(\ln x) \end{aligned}$$

Now differentiate both sides with respect to x, and using the chain and product rule:

$$\begin{aligned}\frac{1}{f(x)} \cdot f'(x) &= \frac{1}{x} \cdot \ln(\ln x) + \ln x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ f'(x) &= f(x) \cdot \left(\frac{1}{x} \cdot \ln(\ln x) + \ln x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}\right) \\ f'(x) &= \boxed{(\ln x)^{\ln x} \cdot \left(\frac{1}{x} \cdot \ln(\ln x) + \ln x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}\right)}\end{aligned}$$

(d) $f(x) = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$.

Solution.

This is a problem where logarithmic differentiation is not necessary, but it is super helpful! We can use it to break up this complicated function in smaller pieces that are easier to take derivatives of.

$$\begin{aligned}\ln[f(x)] &= \ln\left(\frac{x(x-2)}{x^2+1}\right)^{1/3} \\ &= \frac{1}{3}\ln\left(\frac{x(x-2)}{x^2+1}\right) \\ &= \frac{1}{3}\left[\ln(x(x-2)) - \ln(x^2+1)\right] \\ &= \frac{1}{3}\left(\ln(x) + \ln(x-2) - \ln(x^2+1)\right)\end{aligned}$$

Now we can differentiate with respect to x:

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right) \\ f'(x) &= f(x) \cdot \left[\frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)\right] \\ f'(x) &= \boxed{\sqrt[3]{\frac{x(x-2)}{x^2+1}} \cdot \left[\frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)\right]}\end{aligned}$$

Note: if you use a different method for computing the derivative, it is possible that you get an answer that has a completely different form and it is non-trivial to verify that the two answers are actually the same. If you are interested, you can check out the answer Wolframalpha gives. That answer is actually the same as this one.

(e) $u(x) = \sin\left(\tan^{-1}(\ln x)\right)$

Solution. We need to use the chain rule.

$$\begin{aligned} u(x) &= \sin\left(\tan^{-1}(\ln x)\right) \\ u'(x) &= \frac{d}{dx}\left(\boxed{\sin\left(\boxed{\tan^{-1}(\ln x)}\right)}\right) \\ &= \cos\left(\tan^{-1}(\ln x)\right) \cdot \frac{d}{dx}\left(\boxed{\tan^{-1}\left(\boxed{\ln x}\right)}\right) \\ &= \cos\left(\tan^{-1}(\ln x)\right) \cdot \frac{1}{1+(\ln x)^2} \cdot \frac{d}{dx}(\ln x) \\ &= \cos\left(\tan^{-1}(\ln x)\right) \cdot \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x} \end{aligned}$$

(f) $a(t) = \dfrac{(t+2)^3(t+5)^7}{\sqrt{t-5}}$

Solution. This function is the product and quotient of a bunch of functions, and so we should use logarithmic differentiation.

$$a(t) = \frac{(t+2)^3(t+5)^7}{\sqrt{t-5}}$$

We take natural log on both sides:

$$\begin{aligned} \ln(a(t)) &= \ln\left(\frac{(t+2)^3(t+5)^7}{\sqrt{t-5}}\right) \\ &= \ln((t+2)^3(t+5)^7) - \ln(\sqrt{t-5}) \\ &= \ln((t+2)^3) + \ln((t+5)^7) - \ln((t-5)^{1/2}) \\ &= 3\ln(t+2) + 7\ln(t+5) - \frac{1}{2}\ln(t-5) \end{aligned}$$

And now we can differentiate

$$\begin{aligned} \frac{1}{a(t)} \cdot a'(t) &= \frac{3}{t+2} + \frac{7}{t+5} - \frac{1}{2(t-5)} \\ a'(t) &= a(t) \cdot \left(\frac{3}{t+2} + \frac{7}{t+5} - \frac{1}{2(t-5)}\right) \\ &= \frac{(t+2)^3(t+5)^7}{\sqrt{t-5}} \cdot \left(\frac{3}{t+2} + \frac{7}{t+5} - \frac{1}{2(t-5)}\right) \end{aligned}$$

(g) $s(t) = \log_2\left(\log_3\left(e^{t^2}\right)\right)$

Solution. We need the chain rule:

$$s(t) = \log_2\left(\log_3\left(e^{t^2}\right)\right)$$
$$s'(t) = \frac{d}{dt}\left(\boxed{\log_2\left(\boxed{\log_3(e^{t^2})}\right)}\right)$$
$$= \frac{1}{\log_3\left(e^{t^2}\right)\ln 2} \cdot \frac{d}{dt}\left(\boxed{\log_3\left(\boxed{e^{t^2}}\right)}\right)$$
$$= \frac{1}{\log_3\left(e^{t^2}\right)\ln 2} \cdot \frac{1}{e^{t^2}\ln 3} \cdot \frac{d}{dt}\left(\boxed{e^{\boxed{t^2}}}\right)$$
$$= \frac{1}{\log_3\left(e^{t^2}\right)\ln 2} \cdot \frac{1}{e^{t^2}\ln 3} \cdot e^{t^2} \cdot 2t$$

(h) $h(x) = \dfrac{\sqrt{\arctan(4x)}}{e^5}$

Solution. We can think of $h(x)$ as $\frac{1}{e^5}\sqrt{\arctan(4x)}$, and $\frac{1}{e^5}$ is just a constant multiple, so

$$h'(x) = \frac{1}{e^5} \cdot \frac{d}{dx}\left(\boxed{\boxed{\arctan(4x)}^{\frac{1}{2}}}\right)$$
$$= \frac{1}{e^5} \cdot \frac{1}{2} \cdot \arctan(4x)^{-1/2} \cdot \frac{d}{dx}\left(\boxed{\arctan\left(\boxed{4x}\right)}\right)$$
$$= \boxed{\frac{1}{e^5} \cdot \frac{1}{2} \cdot \arctan(4x)^{-1/2} \cdot \left(\frac{1}{1+(4x)^2} \cdot 4\right)}$$

(i) $f(t) = 3^t + t^3 + (\ln t)^{3t}$

Solution. We need to differentiate each piece of the sum, 3^t, t^3, and $(\ln t)^{3t}$. The first and second we can tackle without using logarithmic differentiation; the derivative of 3^t is $3^t \ln 3$, and the derivative of t^3 is $3t^2$. For the third piece, we do need logarithmic differentiation. Let's give this piece a new name, like $g(t)$:

$$g(t) = (\ln t)^{3t}$$

Take the natural log of both sides:

$$\ln g(t) = \ln\left[(\ln t)^{3t}\right]$$
$$= 3t\ln(\ln t)$$

Differentiate both sides:

$$\frac{g'(t)}{g(t)} = 3t \cdot \frac{1}{\ln t} \cdot \frac{1}{t} + 3\ln(\ln t)$$
$$= \frac{3}{\ln t} + 3\ln(\ln t)$$

Solve for $g'(t)$:

$$g'(t) = (\ln t)^{3t}\left[\frac{3}{\ln t} + 3\ln(\ln t)\right]$$

So, $f'(t) = \boxed{3^t \ln 3 + 3t^2 + (\ln t)^{3t}\left[\frac{3}{\ln t} + 3\ln(\ln t)\right]}$.

(j) $f(x) = \log_3(\arctan(x)\arcsin(x))$.

Solution. This needs the chain rule, and then the product rule for the derivative of the inside function:

$$\begin{aligned} f'(x) &= \frac{1}{\arctan(x)\arcsin(x)} \cdot \frac{1}{\ln(3)} \cdot \left[\frac{1}{1+x^2} \cdot \arcsin(x) + \arctan(x) \cdot \frac{1}{\sqrt{1-x^2}}\right] \\ &= \frac{1}{\ln(3)\arctan(x)(1+x^2)} + \frac{1}{\ln(3)\arcsin(x)\sqrt{1-x^2}} \end{aligned}$$

3. *Find the equation of the tangent line to the curve*

$$e^{2x} = \sin(x^2 + 2y) + 1.$$

at the point (0,0).

Solution. Let's start by finding the slope of the tangent line. We know that the slope is just the value of $\frac{dy}{dx}$ at the point $(0,0)$, so let's use implicit differentiation to find this slope.

We start with the equation for the curve:

$$e^{2x} = \sin(x^2 + 2y) + 1$$

Differentiate both sides with respect to x:

$$\begin{aligned} \frac{d}{dx}\left(\boxed{e^{\boxed{2x}}}\right) &= \frac{d}{dx}\left(\boxed{\sin\left(\boxed{x^2+2y}\right)}\right) \\ 2e^{2x} &= \cos(x^2+2y) \cdot \left(2x + 2\frac{dy}{dx}\right) \end{aligned}$$

Plug in $x = 0, y = 0$:

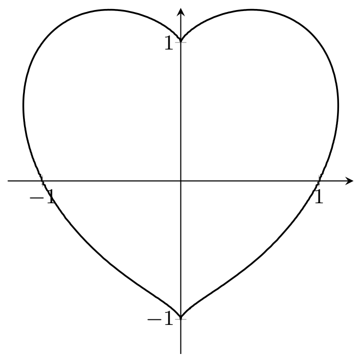
$$2 = 2\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 1$$

So, the line has slope 1. Since $(0,0)$ is a point on the line, the equation of the line is $\boxed{y = x}$.

4. *Find the line tangent to the heart curve* $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$ *at* $(-1, 1)$.



Solution. We start with the equation defining the curve.

$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$$

Differentiate both sides with respect to x:

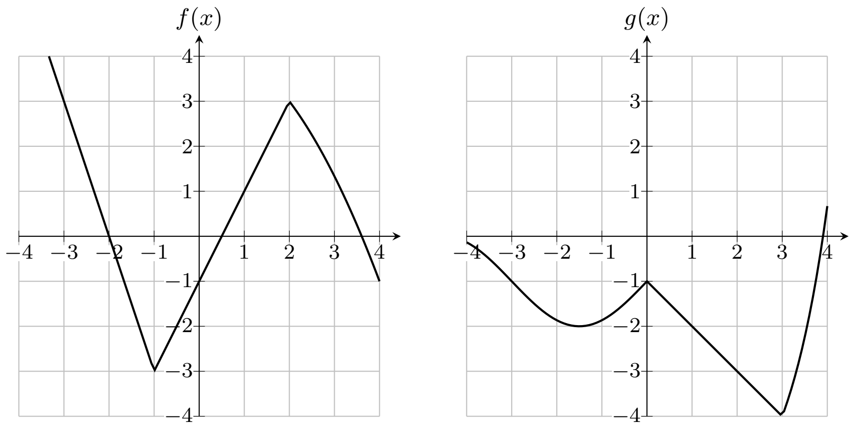
$$\begin{aligned}
\frac{d}{dx}\left(\boxed{\boxed{\left(x^2+y^2-1\right)}^3} - \boxed{x^2}\cdot\boxed{y^3}\right) &= \frac{d}{dx}0 \\
3\left(x^2+y^2-1\right)^2 \cdot \frac{d}{dx}\left(x^2 + \boxed{\boxed{y}^2} - 1\right) - \left[x^2 \cdot \frac{d}{dx}\left(\boxed{\boxed{y}^3}\right) + 2x \cdot y^3\right] &= 0 \\
3(x^2+y^2-1)^2\left(2x + 2y\frac{dy}{dx}\right) - x^2 \cdot 3y^2\frac{dy}{dx} - 2xy^3 &= 0
\end{aligned}$$

Now, we can plug in $x = -1$, $y = 1$:

$$\begin{aligned}
3\left(-2 + 2\frac{dy}{dx}\right) - 3\frac{dy}{dx} + 2 &= 0 \\
3\frac{dy}{dx} - 4 &= 0 \\
\frac{dy}{dx} &= \frac{4}{3}
\end{aligned}$$

Therefore, the slope of the tangent line is $\frac{4}{3}$, so the equation of the line is $\boxed{y - 1 = \frac{4}{3}(x+1)}$. (From the picture, we can see that the tangent line should indeed have positive slope.)

5. *Here are graphs of two functions, $f(x)$ and $g(x)$. If $F(x) = f(g(x))$, what is $F'(1)$?*



Solution. By the Chain Rule, $F'(1) = f'(g(1))g'(1)$. From the graph of g, we see that $g(1) = -2$ and $g'(1) = -1$. Therefore, $F'(1) = -f'(-2)$. From the graph of f, we see that $f'(-2) = -3$, so $F'(1)$ is equal to $\boxed{3}$.

6. *You are given the following information about three functions f, g, and h.*

$$h(1) = 2 \qquad g(2) = 3 \qquad f'(3) = 6$$
$$h'(1) = 4 \qquad g'(2) = 5$$

If $r(x) = f(g(h(x)))$, do you have enough information to find $r'(1)$? If so, compute it. If not, what additional information do you need?

Solution. First, let's find $r'(x)$:

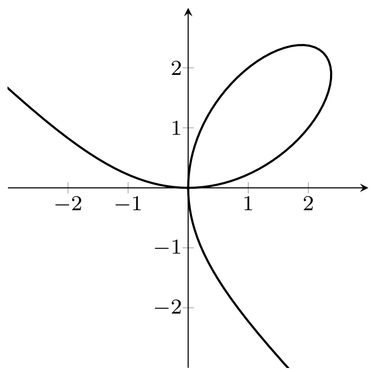
$$\begin{aligned} r'(x) &= \frac{d}{dx}\left(\boxed{f\left(\boxed{g\left(h(x)\right)}\right)}\right) \\ &= f'\left(g\left(h(x)\right)\right) \cdot \frac{d}{dx}\left(\boxed{g\left(\boxed{h(x)}\right)}\right) \\ &= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \end{aligned}$$

Plugging in $x = 1$,

$$\begin{aligned} r'(1) &= f'(g(h(1)))g'(h(1))h'(1) \\ &= f'(g(2))g'(2) \cdot 4 \\ &= f'(3) \cdot 5 \cdot 4 \\ &= 6 \cdot 5 \cdot 4 \\ &= \boxed{120} \end{aligned}$$

(So, we did have enough information to find $r'(1)$.)

7. *Look again at the folium of Descartes, $x^3 + y^3 = \frac{9}{2}xy$. Find all points on the curve where the tangent line is horizontal.*



Solution. We are looking for the points on the curve where $\frac{dy}{dx} = 0$. Starting from the equation for the curve, we can implicitly differentiate with respect to x:

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(\frac{9}{2} \cdot \boxed{x} \cdot \boxed{y}\right)$$
$$3x^2 + 3y^2\frac{dy}{dx} = \frac{9}{2}\left(y + x\frac{dy}{dx}\right)$$

We want points where $\frac{dy}{dx} = 0$:

$$3x^2 + 3y^2 \cdot 0 = \frac{9}{2}\left(y + x \cdot 0\right)$$
$$3x^2 = \frac{9}{2}y$$
$$\frac{2}{3}x^2 = y$$

So, we need $y = \frac{2}{3}x^2$. However, we also need to have points that are actually on the curve, so we also want $x^3 + y^3 = \frac{9}{2}xy$. Plugging the former equation into the latter,

$$x^3 + \left(\frac{2}{3}x^2\right)^3 = \frac{9}{2}x\left(\frac{2}{3}x^2\right)$$
$$x^3 + \frac{8}{27}x^6 = 3x^3$$
$$\frac{8}{27}x^6 - 2x^3 = 0$$
$$x^3\left(\frac{8}{27}x^3 - 2\right) = 0$$

So, either $x = 0$ or $x^3 = \frac{27}{4}$, which happens when $x = \frac{3}{\sqrt[3]{4}}$. Looking at the picture of the curve, we see that the point with $x = 0$ does not have a tangent line at all, but there is a point with a horizontal tangent line, which must be the point with $x = \frac{3}{\sqrt[3]{4}}$. The corresponding y-value is $\frac{2}{3}\left(\frac{3}{\sqrt[3]{4}}\right)^2 = \frac{3}{\sqrt[3]{2}}$. So, the only point on the curve with a horizontal tangent line is $\boxed{\left(\frac{3}{\sqrt[3]{4}}, \frac{3}{\sqrt[3]{2}}\right)}$.

8. Suppose you want to compute derivatives of the following functions. For which would you want to use logarithmic differentiation? Which could you do without logarithmic differentiation? You do not need to actually compute these derivatives!

(a) $(x^3 + 17\sin x)^{\ln 3}$

Solution. We don't need log differentiation here. This function is of the form $[f(x)]^b$ so to compute the derivative, we can use the chain rule!

$$\frac{d}{dx}\left(\boxed{(\boxed{x^3 + 17\sin x})^{\ln 3}} \right) = \ln 3 (x^3 + 17\sin x)^{\ln 3 - 1} \cdot (3x^2 + 17\cos x)$$

(b) $7^{\sqrt{x^3+14}}$

Solution. We don't need log differentiation here. This function is of the form $[b]^{f(x)}$ so to compute the derivative, we can use the chain rule!

$$\begin{aligned}\frac{d}{dx}\left(\boxed{7^{\boxed{\sqrt{x^3+14}}}} \right) &= 7^{\sqrt{x^3+14}} \ln 7 \cdot \frac{d}{dx}\left(\boxed{\sqrt{\boxed{x^3 + 14}}} \right) \\ &= 7^{\sqrt{x^3+14}} \ln 7 \cdot \frac{1}{2}(x^3+14)^{-1/2} \cdot (3x^2)\end{aligned}$$

(c) $(x^2 + \sqrt{x})^{\tan x}$

Solution. Here logarithmic differentiation is *essential* because this function is of the form $f(x)^{g(x)}$. Let $y = (x^2 + \sqrt{x})^{\tan x}$ We will start by taking logs on both side, and then differentiate:

$$\begin{aligned}\ln y &= \ln\left[(x^2 + \sqrt{x})^{\tan x}\right] \\ \ln y &= \tan x \cdot \ln(x^2 + \sqrt{x})\end{aligned}$$

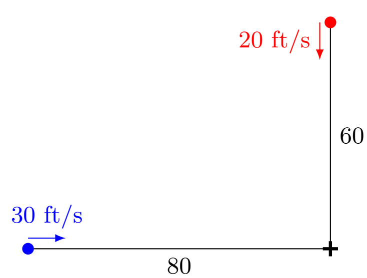
Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(x^2 + \sqrt{x}) + \tan x \cdot \frac{1}{x^2 + \sqrt{x}} \cdot \left(2x + \frac{1}{2}x^{-1/2}\right)$$

9. *Two cars are approaching an intersection. A red car, approaching from the north, is traveling 20 feet per second and is currently 60 feet from the intersection. A blue car, approaching from the west, is traveling 30 feet per second and is currently 80 feet from the intersection. At this moment, is the distance between the two cars increasing or decreasing? How quickly?*

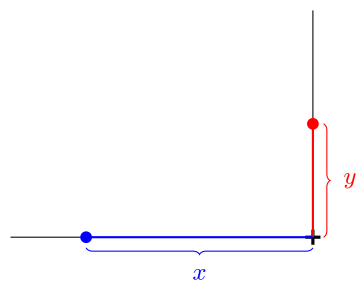
Solution. Intuitively, since both cars are heading toward the intersection, they are getting closer to each other, so the distance between the two cars should be decreasing.

At the moment described in the problem, the situation looks like this (the red and blue car are represented by dots; the intersection is the +):



Snapshot

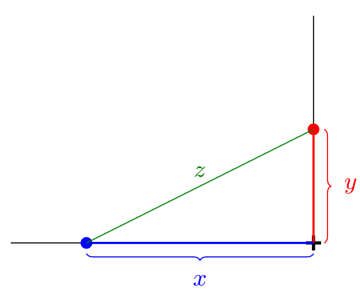
However, the most important picture to draw is the "generic" one that shows how the situation could look at an arbitrary time. Here's a generic picture for this situation. We've labeled some variables in it: x for the blue car's distance from the intersection and y for the red car's distance from the intersection.



Generic

Next, we figure out what we know and what we want to know.

> **What we know:** $\dfrac{dx}{dt} = -30$ (the distance x is *decreasing* at a rate of 30 ft/s) and $\dfrac{dy}{dt} = -20$.
>
> **What we want to know:** Oops! We don't yet have a variable for this. We are asked how quickly the distance between the two cars is decreasing, so let's go back and give a variable name, z, to the distance between the two cars. Then, here is our new picture, and we're looking for $\dfrac{dz}{dt}$.



Now, we try to relate variables. In this case, since the rates we know about are $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and the rate we want to know is $\frac{dz}{dt}$, we try to relate x, y, and z.

We see a right triangle in our picture, so we can use the Pythagorean theorem to relate the lengths of its sides:

> **Equation relating our variables:** $x^2 + y^2 = z^2$

Once we have our relating equation, we differentiate it. Since we are looking for $\frac{dz}{dt}$, we should differentiate it with respect to t.

Now, we differentiate the relating equation with respect to t, using implicit differentiation:

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(z^2)$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \tag{1}$$

Since we care about what happens at a specific moment, we can plug in our "snapshot" information.

At the moment we are interested in, $x = 80$, $y = 60$, $\frac{dx}{dt} = -30$, $\frac{dy}{dt} = -20$, and $z = 100$ by the Pythagorean Theorem. Plugging this all in to the previous equation gives

$$2(80)(-30) + 2(60)(-20) = 2(100)\frac{dz}{dt},$$

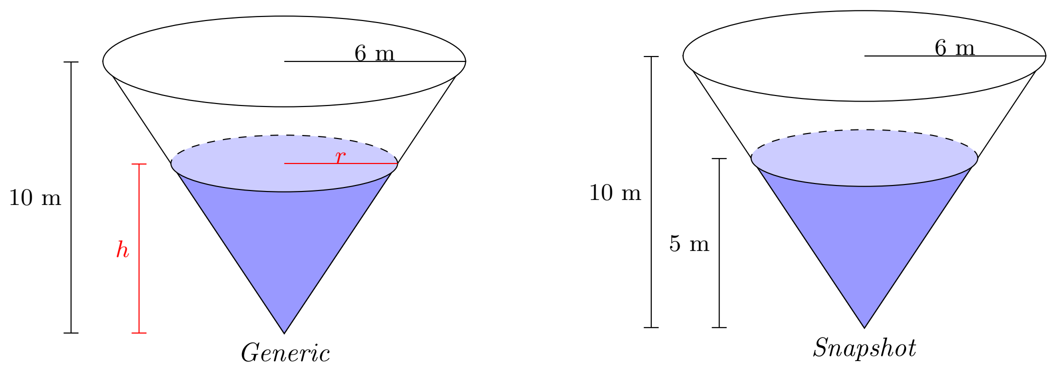
so $\frac{dz}{dt} = -36$. In words, the distance between the cars is

$$\boxed{\text{decreasing at an instantaneous rate of 36 ft/s}}.$$

10. *An oil tank in the shape of an inverted cone has height 10 m and radius 6 m. When the oil is 5 m deep, the tank is leaking oil from the tip at a rate of 2 m^3 per day. How quickly is the height of the oil in the tank decreasing at this moment?*

    Note: The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

    Solution. As usual, let's start by drawing two pictures, a generic picture that applies at any time, and a "snapshot" showing the situation at the time we care about. We'll also label some variables in the generic picture (the things labeled in red are changing).



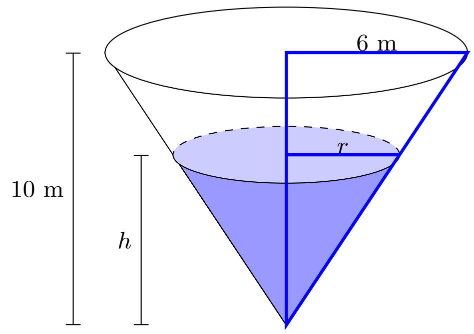
Note that the radius and height of the tank are constant, but the radius and height of the oil change as oil leaks out, so we assign variables to those.

Next, we figure out what we know and what we want to know.

> **What we know:** If we let V represent the volume of oil (in cubic m) at time t (where t is measured in days), then $\frac{dV}{dt} = -2$. (This rate is negative because the volume is decreasing.)
>
> **What we want to know:** $\frac{dh}{dt}$

Now, we try to relate variables. In this case, since the rate we know is $\frac{d\boldsymbol{V}}{dt}$ and the rate we want to know is $\frac{d\boldsymbol{h}}{dt}$, we try to relate V to h.

First, the volume V of oil is the volume of a cone with radius r and height h, so $V = \frac{1}{3}\pi r^2 h$. We'd like to get V in terms of just h, so we need to express r in terms of h. Here, the key is similar triangles:



From the similar triangles shown in blue, we get $\dfrac{6}{10} = \dfrac{r}{h}$, so $r = \dfrac{3h}{5}$. Plugging this into our equation for V, we get

Equation relating our variables: $V = \dfrac{1}{3}\pi\left(\dfrac{3h}{5}\right)^2 h = \dfrac{3\pi}{25}h^3$

Once we have our relating equation, we differentiate it. Since we are looking for $\frac{dh}{d\boldsymbol{t}}$, we should differentiate with respect to t.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{3\pi}{25}h^3\right)$$
$$\frac{dV}{dt} = \frac{9\pi}{25}h^2\frac{dh}{dt}$$

At the moment we're interested in, $\frac{dV}{dt} = -2$ and $h = 5$:

$$-2 = \frac{9\pi}{25}(25)\frac{dh}{dt}$$

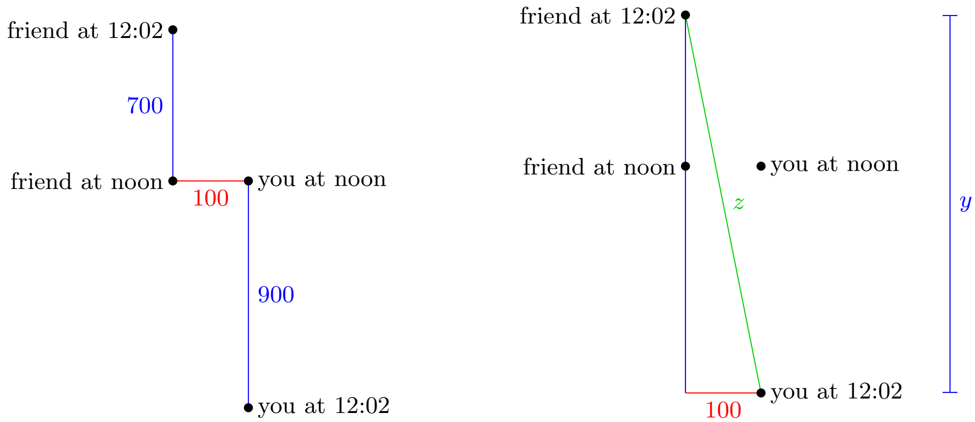
So, $\dfrac{dh}{dt} = -\dfrac{2}{9\pi}$. So, at the time we're interested in, the height of oil is

$$\boxed{\text{decreasing at } \tfrac{2}{9\pi} \text{ m/day}}$$

11. *At noon, you are running to get to class and notice a friend 100 feet west of you, also running to class. If you are running south at a constant rate of 450 ft/min (approximately 5 mph) and your friend is running north at a constant rate of 350 ft/min (approximately 4 mph), how fast is the distance between you and your friend changing at 12:02 pm?*

    Solution. As usual, let's start by drawing two pictures, a generic picture that applies at any time, and a "snapshot" showing the situation at the time we care about. We'll also label some variables in the generic picture.

    Between noon and 12:02, you run (450 ft/min)(2 min) = 900 ft, and your friend runs (350 ft/min)(2 min) = 700 ft.



Snapshot (12:02) *Generic*

In the generic picture, we've let z be the distance (in feet) between you and your friend and y be the "vertical" (north-south) distance (in feet) between you and your friend.

Next, we figure out what we know and what we want to know.

What we know: $\frac{dy}{dt} = 450 + 350 = 800$ ft/min

What we want to know: $\frac{dz}{dt}$

Now, we try to relate variables. In this case, since the rates we know is $\frac{dy}{dt}$ and the rate we want to know is $\frac{dz}{dt}$, we try to relate y and z. Using our generic picture, we see that we can relate our variables using the Pythagorean Theorem:

Equation relating our variables: $y^2 + 100^2 = z^2$

Once we have our relating equation, we differentiate it. Since we are looking for $\frac{dz}{dt}$, we should differentiate with respect to t.

$$\frac{d}{dt}\left(y^2 + 100^2\right) = \frac{d}{dt}\left(z^2\right)$$
$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

We can simplify this a bit by dividing through by 2:

$$y\frac{dy}{dt} = z\frac{dz}{dt}$$

Now, we plug in the snapshot information and rate we know. To get z at the snapshot moment, we can use the Pythagorean Theorem (or the relation $y^2 + 100^2 = z^2$ that we got from using the Pythagorean Theorem earlier):

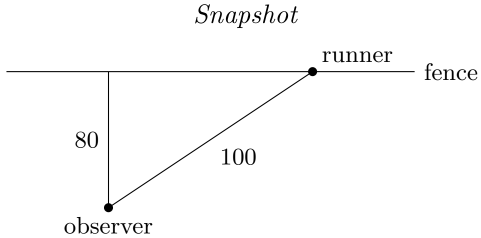
$$1600(800) = \sqrt{1600^2 + 100^2}\frac{dz}{dt}$$

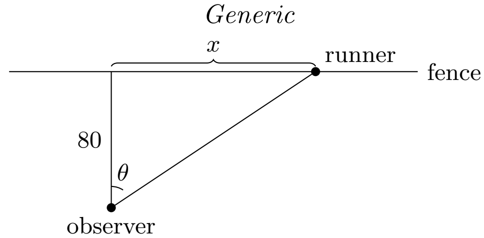
So, $\frac{dz}{dt} = \frac{12800}{\sqrt{257}}$; that is, the distance between you and your friend is $\boxed{\text{increasing at } \frac{12800}{\sqrt{257}} \text{ ft/min}}$ at 12:02.

12. During a night run, an observer is standing 80 feet away from a long, straight fence when she notices a runner running along it, getting closer to her. She points her flashlight at him and keeps it on him as he runs.

When the distance between her and the runner is 100 feet, he is running at 9 feet per second. At this moment, at what rate is she turning the flashlight to keep him illuminated? Include units in your answer.

Solution. As usual, let's start by drawing two pictures, a generic picture that applies at any time, and a "snapshot" showing the situation at the time we care about. We'll also label some variables in the generic picture.





Next, we figure out what we know and what we want to know.

What we know: $\frac{dx}{dt} = -9$ (negative because the man is getting closer to the observer, so the distance x is decreasing)

What we want to know: $\frac{d\theta}{dt}$

Now, we try to relate variables. In this case, since the rate we know is $\frac{dx}{dt}$ and the rate we want to know is $\frac{d\theta}{dt}$, we try to relate x and θ.

Equation relating our variables: $\tan\theta = \frac{x}{80}$

Once we have our relating equation, we differentiate it. Since we are looking for $\frac{d\theta}{dt}$, we should differentiate with respect to t.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{80}\right)$$
$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{80}\frac{dx}{dt}$$

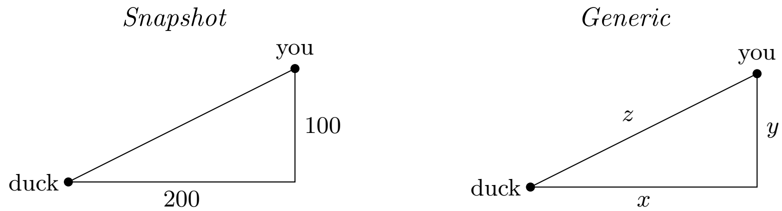
Now, we plug in our snapshot information. In our snapshot, we can see that $\cos\theta = \frac{80}{100} = \frac{4}{5}$, so $\sec\theta = \frac{5}{4}$:

$$\frac{25}{16}\frac{d\theta}{dt} = \frac{1}{80}(-9)$$
$$\frac{d\theta}{dt} = -\frac{9}{80}\cdot\frac{16}{25}$$
$$= -\frac{9}{125}$$

So, the observer must turn the flashlight at $\boxed{\frac{9}{125} \text{ radians per second}}$.

13. As you're riding up an elevator, you spot a duck on the ground, waddling straight towards the base of the elevator. The elevator is rising at a speed of 10 feet per second, and the duck is moving at 5 feet per second towards the base of the elevator. As you pass the eighth floor, 100 feet up from the level of the river, the duck is 200 feet away from the base of the elevator. At this instant, at what rate is the distance between you and the duck changing?

    Solution. As usual, let's start by drawing two pictures, a generic picture that applies at any time, and a "snapshot" showing the situation at the time we care about. We'll also label some variables in the generic picture.

    Next, we figure out what we know and what we want to know.

    What we know: $\frac{dx}{dt} = -5$ (negative because x is decreasing) and $\frac{dy}{dt} = 10$ (positive because y is increasing)

    What we want to know: $\frac{dz}{dt}$

    Now, we try to relate variables. In this case, since the rates we know about are $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and the rate we want to know is $\frac{dz}{dt}$, we try to relate x, y, and z.

    Equation relating our variables: $x^2 + y^2 = z^2$

    Once we have our relating equation, we differentiate it. Since we are looking for $\frac{dz}{dt}$, we should differentiate with respect to t.

    $$\frac{d}{dt}\left(x^2+y^2\right) = \frac{d}{dt}\left(z^2\right)$$
    $$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

    Let's divide by 2 to simplify the equation a little:

    $$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

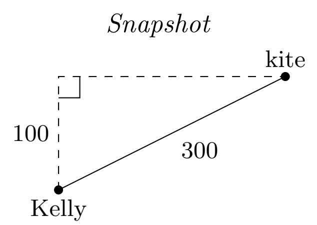
    Plug in the snapshot information:

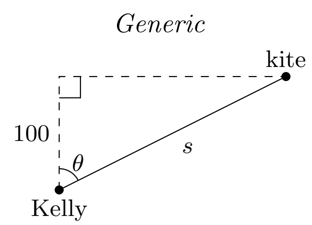
    $$200\,(-5) + 100(10) = \sqrt{200^2+100^2}\,\frac{dz}{dt}$$

    Solving, $\frac{dz}{dt} = \boxed{0 \text{ ft/s}}$.

14. *Kelly is flying a kite; the kite is 100 ft above the ground and moving horizontally away from Kelly. At precisely 1 pm, Kelly has let out 300 ft of string, and the amount of string let out is increasing at a rate of 5 ft/s. If Kelly is standing still, at what rate is the angle between the string and the vertical increasing at 1 pm? (You may assume that the string is stretched taut so that it is a straight line.)*

Solution. As usual, let's start by drawing two pictures, a generic picture that applies at any time, and a "snapshot" showing the situation at the time we care about. We'll also label some variables in the generic picture.





Note that, since the kite flies horizontally, it is always 100 ft above the ground; that's why we can label the height as 100 in the generic picture.

Next, we figure out what we know and what we want to know.

What we know: $\frac{ds}{dt} = 5$ (positive because the amount of string is increasing)

What we want to know: $\frac{d\theta}{dt}$

Now, we try to relate variables. In this case, since the rate we know is $\frac{ds}{dt}$ and the rate we want to know is $\frac{d\theta}{dt}$, we try to relate s and θ.

Equation relating our variables: $\cos\theta = \frac{100}{s}$

Once we have our relating equation, we differentiate it. Since we are looking for $\frac{d\theta}{dt}$, we should differentiate with respect to t.

$$\begin{aligned}\frac{d}{dt}(\cos\theta) &= \frac{d}{dt}\left(100s^{-1}\right)\\ -\sin\theta\frac{d\theta}{dt} &= -100s^{-2}\frac{ds}{dt}\end{aligned}$$

We plug in our snapshot information. By the Pythagorean Theorem, the width of the triangle in our snapshot is $100\sqrt{8} = 200\sqrt{2}$, so $\sin\theta = \frac{200\sqrt{2}}{300} = \frac{2\sqrt{2}}{3}$ in our snapshot:

$$\begin{aligned}-\frac{2\sqrt{2}}{3}\frac{d\theta}{dt} &= -\frac{100}{300^2}\cdot 5\\ \frac{d\theta}{dt} &= \frac{3}{2\sqrt{2}}\cdot\frac{500}{300^2}\\ &= \boxed{\frac{1}{120\sqrt{2}}\text{ radians per second}}\end{aligned}$$

15. *A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at a rate of 5 ft/sec.*

    (a) *How fast is the top of the ladder sliding down the wall then?*

    (b) *At what rate is the area of the triangle formed by the ladder, wall and ground changing then?*

(c) *At what rate is the angle between the ladder and the ground changing then?*

Solution. Caveat: We really want you to follow the problem solving framework that has been set up in the previous problems as you work through these. This solution is a sketch, and you should work to fill in the details you would need in order to get full credit on this problem!

a) Denote the base of the triangle by x, and the height of the triangle by y. We have $x^2 + y^2 = 13^2$. Taking derivative with respect to time yields $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. At the moment we are interested in, we know that $\frac{dx}{dt} = 5$, $x = 12$ and $y = \sqrt{13^2 - 12^2} = 5$, so $2 \cdot 12 \cdot 5 + 2 \cdot 5\frac{dy}{dt} = 0$. So $\frac{dy}{dt} = -12$ ft/sec.

b) Denote the area of the triangle by A. We have $A = \frac{xy}{2}$. So $\frac{dA}{dt} = \frac{\frac{dx}{dt}y + x\frac{dy}{dt}}{2}$. Plugging in everything we know from the previous part, $\frac{dA}{dt} = \frac{5 \cdot 5 + 12 \cdot (-12)}{2} = -59.5\ \text{ft}^2/\text{sec}$.

c) Denote the angle by θ. We will use the equation $\tan\theta = \frac{y}{x}$. After applying $\frac{d}{dt}$, we have $\sec^2(\theta)\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$. At the moment we are interested in, we have $\cos(\theta) = \frac{12}{13}$. So $\frac{d\theta}{dt} = \frac{12^2}{13^2}\frac{-12*12-5*5}{12^2} = \frac{-169}{13^2} = -1\frac{rad}{sec}$. (Side note: The angle must be measured is radians for this computation to work! If we used degrees instead of radians, we would have to change our trig derivative rule to $[\tan\theta]' = \frac{\pi}{180}\sec^2(\theta)$.)

16. (a) *Use linear approximation to estimate $\sqrt{24.5}$ without using a calculator. Draw a sketch to explain what you are doing.*

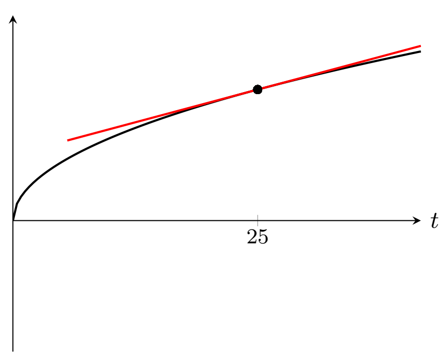
Solution. We need to first pick a function to approximate; let's use $f(x) = \sqrt{x}$. We want to approximate $f(24.5)$, so we should pick a value of a near 24.5 as a "base" for our approximation. Let's use $a = 25$, since we can evaluate $f(25)$ exactly.

$f'(x) = \frac{1}{2}x^{-1/2}$, so $f'(a) = \frac{1}{10}$. Thus, our tangent line has slope 10. We also know that $(25, 5)$ is a point on the tangent line, so the line has equation $y - 5 = \frac{1}{10}(x - 25)$, or $y = 5 + \frac{1}{10}(x - 25)$.

Therefore, $\sqrt{x} \approx 5 + \frac{1}{10}(x - 25)$ for x near 25.

Plugging in $x = 24.5$ gives $\sqrt{24.5} \approx 5 + \frac{1}{10}(-0.5) = 4.95$.

Here's a picture of our situation:



(b) *From your sketch, you should be able to tell whether your approximation is an overestimate or underestimate. Which is it?*

Solution. From the sketch it is clear that our estimate is an overestimate.

(c) *Explain your answer to (b) using a second derivative.*

Solution. As we've seen, it is the concavity of a function that decides whether linear approximations give over-estimates or under-estimates. So far, we've been able to "see" for ourselves whether a function is concave up or down. But there is a more mathematical way to decide this. Recall our old table again:

f	f'	f''
Positive / Negative	nothing	—
increasing / decreasing	Positive / Negative	nothing
concave up / concave down	increasing / decreasing	Positive / Negative
—	concave up / concave down	increasing / decreasing

A function f is concave up exactly when the first derivative is increasing i.e. exactly when the second derivative is positive. So an easy way to check concavity, is simply to compute the second derivative. Here, $f''(x) = \frac{-1}{4}x^{-3/2} = \frac{-1}{4} \cdot \frac{1}{\sqrt{x^3}}$. Since this is negative for any value of x, the function is concave down, and so the estimate is an over-estimate.

17. *Use linear approximation to estimate $9^{4/3}$. Is your estimate too high or too low?*

Solution. The idea of linear approximation is that the line tangent to a graph $y = f(x)$ at $x = a$ is a good approximation for the actual graph around $x = a$, so we start by picking a function $f(x)$ and a value of x where we will find the tangent line. Since we are looking for $9^{4/3}$, it makes sense to pick as our function $f(x) = x^{4/3}$. Then, we want to know $f(9)$. Let's find the tangent line to $f(x)$ at $x = 8$ because 8 is pretty close to 9, and we can evaluate $f(8)$ exactly; it's $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$.

Next, we find the tangent line. The slope of the tangent line will be given by $f'(8)$. Using the Power Rule, $f'(x) = \frac{4}{3}x^{1/3}$, so $f'(8) = \frac{4}{3} \cdot 8^{1/3} = \frac{8}{3}$. Thus, the tangent line has slope $\frac{8}{3}$; we know $(8, 16)$ is a point on the tangent line, so the tangent line has equation $y - 16 = \frac{8}{3}(x - 8)$, or $y = 16 + \frac{8}{3}(x - 8)$.

So, we know that $x^{4/3} \approx 16 + \frac{8}{3}(x - 8)$ for x near 8. Plugging in $x = 9$ gives $9^{4/3} \approx 16 + \frac{8}{3} = \boxed{18 + \frac{2}{3}}$.

To figure out whether this estimate is too high or too low, let's think about the concavity of $f(x)$. The second derivative $f''(x)$ is $\frac{4}{9}x^{-2/3}$, which is positive on $(0, \infty)$; therefore, f is concave up on $(0, \infty)$. So, our tangent line approximation must have been under the graph of f, which means that our estimate was $\boxed{\text{too low}}$.

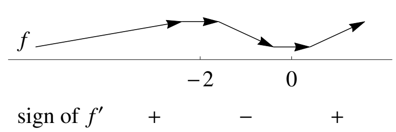
18. In this problem, we'll look at the cubic function $f(x) = x^3 + 3x^2 + 1$.

(a) *Find all critical points of f.*

Solution. $f'(x) = 3x^2 + 6x = 3(x^2 + 2x) = 3x(x + 2)$, so $f'(x) = 0$ at $x = \boxed{-2, 0}$. (There are no points where f' is undefined.)

(b) *Make a sign chart for f', and use this to decide whether each of the critical points you found is a local minimum, a local maximum, or neither.*

Solution.



So we know that at $x = -2$ there is a local max and at $x = 0$ there is a local minimum. The above is a strategy used to classify critical point. The strategy can be boiled down to the following:

First Derivative Test. Suppose $f'(c) = 0$ or $f'(x)$ doesn't exist at $x = c$.

- If f' changes sign from negative to positive at $x = c$, then f has a local minimum at c.
- If f' changes sign from positive to negative at $x =$, then f has a local maximum at c.

19. Let $f(x) = x^4 - 8x^2 + 16$.

(a) *Find all critical points of f.*

Solution. To find the critical points, we need to know when the derivative equals 0.

$$f'(x) = 4x^3 - 16x$$

$$f'(x) = 4x(x^2 - 4) = 0$$

So the critical points are

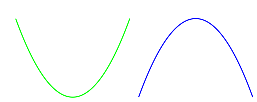
$$x = -2,\ 0, \text{and } 2$$

(b) *For each critical point c of f, find the sign of $f''(c)$. What does this tell you about the critical point c?*

Solution. $f''(x) = 12x^2 - 16$

x	$f''(x)$
0	-16
2	32
-2	32

Note that a critical point on a concave up portion of the graph is a local min and a critical point on a concave down portion of the graph is a local max.



20. Suppose you've made the following sign chart for the derivative of a function $f(x)$. The function $f(x)$ is continuous and differentiable on $(-\infty, \infty)$.

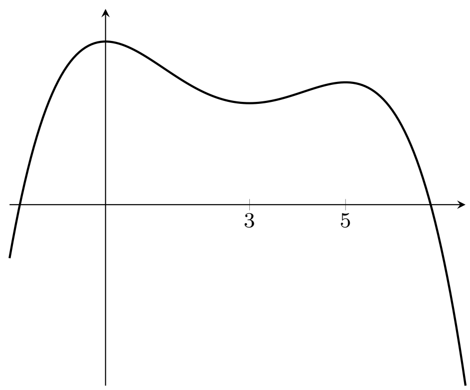
		0		3		5	
sign of f'	$+$		$-$		$+$		$-$

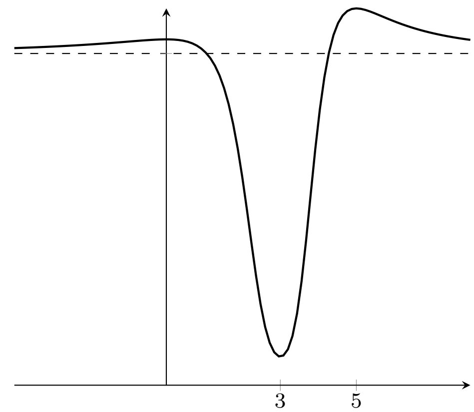
(a) *Does $f(x)$ have an absolute maximum on $(-\infty, \infty)$? (Definitely yes, definitely no, or maybe?) If so, where could it be?*

Solution. The function definitely has an absolute maximum at either $x = 0$ or at $x = 5$.

(b) *Does $f(x)$ have an absolute minimum on $(-\infty, \infty)$? If so, where could it be?*

Solution. There is not enough information to be sure. For example, it's possible that $\lim_{x\to\infty} f(x) = -\infty$, in which case f certainly would not have an absolute minimum. However, it's also possible that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist and that $x = 3$ is an absolute minimum point. Both possibilities are illustrated below:





(c) *Does $f(x)$ have an absolute maximum on $[-2, 10]$? If so, where could it be?*

Solution. Yes, f must have an absolute maximum on $[-2, 10]$; it could be at $x = 0$ or $x = 5$.

(d) *Does $f(x)$ have an absolute minimum on $[-2, 10]$? If so, where could it be?*

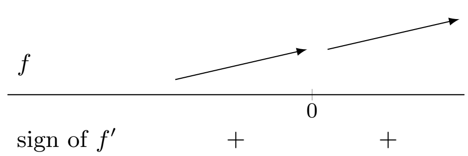
Solution. Yes, f must have an absolute minimum on $[-2, 10]$; it could be at $x = -2$, $x = 3$, or $x = 10$.

21. *Let $f(x) = 3x^{1/3} + 4x$. Find all critical points of f, and determine whether each critical point is a local minimum, local maximum, or neither.*

Solution. To find the critical points of $f(x)$, we need to look for where the derivative is 0 or where the derivative is undefined.

$$f'(x) = x^{-\frac{2}{3}} + 4$$

it's never 0 because $x^{-2/3} = \left(x^{-1/3}\right)^2$ is always ≥ 0.(1) Note that $f'(x)$ is undefined at $x = 0$. This means that there is a critical point at $x = 0$. The Second Derivative Test isn't useful here: $f''(x) = -\frac{2}{3}x^{-5/3}$, so $f''(0)$ is undefined (which makes sense since we already found that f is not differentiable at 0). Instead, we'll use the First Derivative Test; for example, $f'(-1) = 5 > 0$ and $f'(1) = 5 > 0$. That gives us the following sign chart for f':



So, **the critical point $x = 0$ is neither a local minimum nor a local maximum**.

(1) You could also try solving the equation $f'(x) = 0$, which would give:

$$\begin{aligned} x^{-2/3} &= -4 \\ x^{2/3} &= -\frac{1}{4} \\ x &= \left(-\frac{1}{4}\right)^{3/2} \\ &= \sqrt{\left(-\frac{1}{4}\right)^3}, \end{aligned}$$

which is not a real number.

22. *Sanity Check. Do this without looking at your notes! Which of the following is the linearization of a function $f(x)$ at $x = a$?*

A. $y = f(a) + f'(x)(x - a)$

B. $y = f(x) + f'(a)(x - a)$

C. $y = f(x) + f'(x)(x - a)$

(*D.*) $y = f(a) + f'(a)(x - a)$

E. $y = f'(a) + f(x)(x - a)$

F. $y = f'(x) + f(a)(x - a)$

G. $y = f'(x) + f(x)(x - a)$

H. $y = f'(a) + f(a)(x - a)$

23. (a) *Use linear approximation to approximate the value of* $\cos\left(\frac{93}{180}\pi\right)$ *and* $\cos\left(\frac{86}{180}\pi\right)$.

Solution. Our goal is to use linear approximation to get a sense of the value of $f(x) = \cos(x)$ at some unfamiliar angles. Note that both the angles are close to $x = \frac{\pi}{2}$, which is helpful since we know that $\cos(\frac{\pi}{2}) = 0$. This observation sharpens our goal; we want to find the equation of the tangent line at $(\frac{\pi}{2}, 0)$. The slope of the tangent line will be equal to $f'(\frac{\pi}{2})$.

$$f'(x) = -\sin(x)$$

$$f'(\frac{\pi}{2}) = -1$$

As a sense making check that we wrote down the correct derivative, we see that $\cos(x)$ is decreasing on the interval $(0, \frac{\pi}{2})$ and so the negative sign is required.

Using the point slope formula we can write down the equation of the tangent line

$$y = -1\left(x - \frac{\pi}{2}\right)$$

The punchline is that the tangent line is a good approximation for $\cos(x)$ for values near $\frac{\pi}{2}$:

$$\cos(x) \approx -x + \frac{\pi}{2} \text{ for } x \text{ near } \frac{\pi}{2}.$$

$$\cos\left(\frac{93}{180}\pi\right) \approx -\frac{93}{180}\pi + \frac{\pi}{2} = -\frac{3}{180}\pi = \frac{-\pi}{60}$$

$$\cos\left(\frac{86}{180}\pi\right) \approx -\frac{86}{180}\pi + \frac{\pi}{2} = \frac{4}{180}\pi = \frac{\pi}{45}$$

(b) *To determine if the estimates are over or under the actual values, it is helpful to determine the concavity of the function. On the interval* $[0,\pi]$ *where is* $f(x) = \cos(x)$ *concave up? On the interval* $[0,\pi]$ *where is* $\cos(x)$ *concave down? Justify your answer using calculus.*

Solution. The second derivative measures concavity; to decide if a function is concave up or concave down we need to make use of the second derivative.

$$f'(x) = -\sin(x)$$

$$f''(x) = -\cos(x)$$

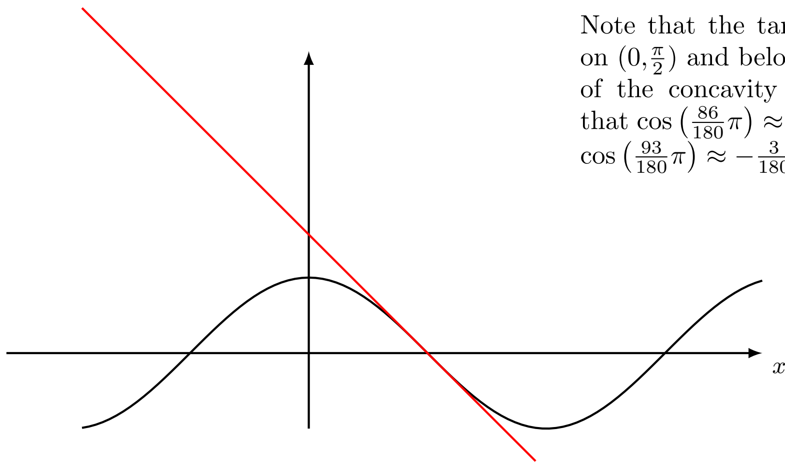
The function is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$. To decide the sign of $f''(x)$, we solve $f''(x) = 0$.

$$-cos(x) = 0$$
$$x = \frac{\pi}{2}$$

- $f''(x)$ is negative on the interval $(0, \frac{\pi}{2})$ and so $f(x)$ is concave down.
- $f''(x)$ is positive on the interval $(\frac{\pi}{2}, \pi)$ and so $f(x)$ is concave up.

(c) *Are your estimates overestimates or underestimates? Sketch a graph to justify.*

Solution. We will graph $\cos(x)$ along with the tangent line at $(\frac{\pi}{2}, 0)$ to decide if our estimates are overestimates or underestimates.



Note that the tangent line is above the curve on $(0,\frac{\pi}{2})$ and below the curve on $(\frac{\pi}{2},\pi)$ because of the concavity we discovered. This shows that $\cos\left(\frac{86}{180}\pi\right) \approx \frac{4}{180}\pi$ is an overestimate while $\cos\left(\frac{93}{180}\pi\right) \approx -\frac{3}{180}\pi$ is an underestimate.

24. *Use linear approximation to estimate $e^{0.1}$. Explain whether your estimate is overestimate of under estimate.*

Solution. Let $f(x) = e^x$. We need to use an appropriate tangent line. Since 0 is near 0.1, and we know that $e^0 = 1$, let's consider the tangent line at $x = 0$. Since $f(0) = 1$, this tangent line passes through the point (0,1). We also need the slope at $x = 1$:

$$f'(x) = e^x,$$

and so $f'(0) = 1$. Then the equation of the tangent line is

$$y = x + 1.$$

Then $e^x \approx x + 1$ for x near 0. And so

$$e^{0.1} = f(0.1) \approx 0.1 + 1 = 1.1.$$

To find out whether this estimate is over or under estimate, we compute the second derivative and see that

$$f''(x) = e^x,$$

This is always positive, and so $f(x)$ is concave up. That is, the estimate is an underestimate.

25. *Use linear approximation to estimate* $\sin^2(\frac{\pi}{4} - 0.1)$.

Solution. Let $f(x) = \sin^2(x)$. By the chain rule, $f'(x) = 2\sin(x)\cos(x)$. Using $a = \frac{\pi}{4}$, we have $f(x) \approx f(a) + f'(a)(x-a)$ near $x = a$. Since $f(a) = (\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$ and $f'(a) = 2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 1$, we have $f(x) \approx \frac{1}{2} + (x - \frac{\pi}{4})$ near $x = \frac{\pi}{4}$. Thus $f(\frac{\pi}{4} - 0.1) \approx \frac{1}{2} + (\frac{\pi}{4} - 0.1 - \frac{\pi}{4}) = 0.4$.

26. We'd like to get an accurate graph of $f(x) = \dfrac{\pi x^2 + 2}{x}$. Answer the questions below in whatever order makes most sense to you, and use them to sketch the graph of f.

(a) *What is the domain of f? Does f have any useful symmetry?*

Solution. The domain of f is all real numbers except $x = 0$. That is the domain is $\boxed{(-\infty,0) \cup (0,\infty)}$. To see whether f has any useful symmetry, we look at $f(-x)$:

$$\begin{aligned} f(-x) &= \frac{\pi(-x)^2 + 2}{-x} \\ &= \frac{-\pi x^2 + 2}{x} \\ &= -f(x) \end{aligned}$$

So f is $\boxed{\text{odd}}$. This is great! In order to graph f, we can really analyze it on just $(0, \infty)$ and then use symmetry to fill in the other half of the graph.

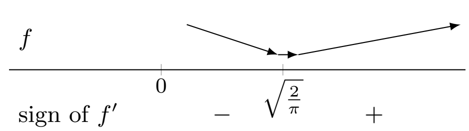
Let's do that now, and then we'll answer the remaining questions using our graph.

- Intercepts: Since $x \neq 0$, there is $\boxed{\text{no } y\text{-intercept}}$. Since $\pi x^2 + 2 \neq 0$ for any x, there are $\boxed{\text{no } x\text{-intercepts}}$. So this graph never crosses the x or y axes.

- Discontinuities: f has a discontinuity at $x = 0$. We classify discontinuities using limits. Let's check to see what sort of discontinuity occurs at $x = 0$. We want to compute $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{\pi x^2 + 2}{x}$. This limit is of the form $\left(\dfrac{2}{0}\right)$. Since the denominator is tiny positive as $x \to 0^+$, we have that $f(x) \to \infty$ as $x \to 0^+$. That is, there is a $\boxed{\text{vertical asymptote at } x = 0}$.

- Horizontal Asymptotes: To find the horizontal asymptote on $(0,\infty)$, we need to compute the limit $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{\pi x^2 + 2}{x} = \lim_{x \to \infty} \left(\pi x + \dfrac{2}{x}\right) = \infty$. So there are $\boxed{\text{no horizontal asymptotes}}$.

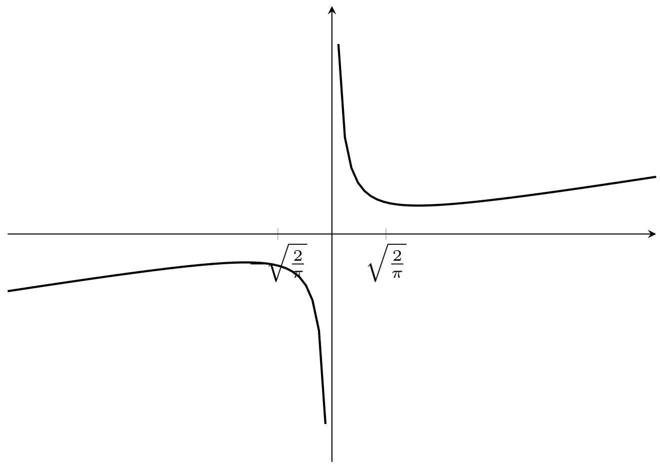
- Where is f increasing or decreasing? To figure out where f is increasing, we'll make a sign chart for $f'(x) = \pi - \frac{2}{x^2} = \frac{\pi x^2 - 2}{x^2}$. f' is not defined at $x = 0$, and it's 0 when $\pi x^2 = 2 \Rightarrow \pm\sqrt{\dfrac{2}{\pi}}$.
We will only focus on $x = 0, \sqrt{\dfrac{2}{\pi}}$ since we are only considering the right half of the graph. By testing points, we get the following sign chart for f' on $(0, \infty)$:



For example, $f'(0.5) < 0$ and $f'(100) > 0$.

- Where is f concave up or concave down? We'll figure out the concavity of f by making a sign chart for $f''(x) = \frac{4}{x^3}$. This is undefined at $x = 0$, and its always positive on $(0, \infty)$. That is f is $\boxed{\text{concave up on } (0,\infty)}$.

Putting this all together, we can get a reasonable graph of f on $(0, \infty)$ and then use symmetry to draw the graph on $(-\infty, 0)$:



Now that we've analyzed and graphed f, it's easy to answer the remaining questions.

(b) *Find and classify all discontinuities of f. Justify your classifications using limits.*

Solution. There is a vertical asymptote at $x = 0$.

(c) *Find all horizontal asymptotes of f.*

Solution. There are no horizontal asymptotes.

(d) *On what intervals is $f(x)$ increasing? On what intervals is it decreasing?*

Solution. f is increasing on $\left(-\infty, -\sqrt{\frac{2}{\pi}}\right) \cup \left(\sqrt{\frac{2}{\pi}}, \infty\right)$ and decreasing on $\left(-\sqrt{\frac{2}{\pi}}, 0\right) \cup \left(0, \sqrt{\frac{2}{\pi}}\right)$

(e) *On what intervals is $f(x)$ concave up? On what intervals is it concave down?*

Solution. f is concave up on $(0,\infty)$ and concave down on $(-\infty,0)$.

(f) *What are the inflection points of f?*

Solution. Although concavity changes at $x = 0$, there isn't an inflection point there because f is undefined at 0.

(g) *Can you find the x- and y-intercepts of f?*

Solution. We found that there are no x and y intercepts. The graph never crosses the axes.

(h) *Sketch a rough graph of $f(x)$ that incorporates all of the above information.*

Solution. See graph above.

27. Let $f(x) = 3x^{5/3}(x-8)$.

Answer the questions below in whatever order makes most sense to you, and use them to sketch the graph of f.

(a) *What is the domain of f? Does f have any useful symmetry?*

Solution. $\boxed{\text{The domain of } f \text{ is all real numbers.}}$ To see whether f has any useful symmetry, we look at $f(-x)$:

$$\begin{aligned} f(-x) &= 3(-x)^{5/3}(-x-8) \\ &= -3x^{5/3}(-x-8) \end{aligned}$$

This is not equal to $f(x)$ or $-f(x)$, so f is $\boxed{\text{neither even nor odd}}$.

(b) *Find and classify all discontinuities of f. Justify your classifications using limits.*

Solution. Since f is made up of familiar functions, its discontinuities are exactly where it is undefined. However, we've already said that f is defined everywhere, so f has $\boxed{\text{no discontinuities}}$.

(c) *Find all horizontal asymptotes of f.*

Solution. To find the horizontal asymptotes, we must evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. As $x \to \infty$, both $x^{5/3}$ and $x-8$ increase without bound, so $\boxed{\lim\limits_{x\to\infty} 3x^{5/3}(x-8) = \infty}$.

As $x \to -\infty$, both $x^{-5/3}$ and $x-8$ decrease without bound, so $\boxed{\lim\limits_{x\to\infty} 3x^{5/3}(x-8) = \infty}$.

(d) *On what intervals is $f(x)$ increasing? On what intervals is it decreasing?*

Solution. To answer this, let's make a sign chart for f', since the sign of f' tells us whether f is increasing or decreasing. Using the Product Rule,

$$\begin{aligned} f'(x) &= (5x^{2/3})(x-8) + 3x^{5/3} \\ &= 8x^{5/3} - 40x^{2/3} \\ &= 8x^{2/3}(x-5) \end{aligned}$$

The critical points, where $f'(x)$ is 0 or undefined, are $x = 0, 5$. Now, we can make a sign chart for f' by testing points. For example, $f'(-1) = 8(1)(-6) < 0$, $f'(1) = 8(1)(-4) < 0$, and $f'(6) = 8(6^{2/3})(1) > 0$. So, here is the sign chart for f':

		0		5	
sign of f'	$-$		$-$		$+$

So, $\boxed{f \text{ is decreasing on } (-\infty, 5) \text{ and increasing on } (5, \infty)}$. (It's also okay to say that f is decreasing on $(-\infty, 0)$ and $(0, 5)$.)

(e) *On what intervals is $f(x)$ concave up? On what intervals is it concave down?*

Solution. To answer this, let's make a sign chart for f'', since the sign of f'' tells us whether f is increasing or decreasing. We found that $f'(x) = 8x^{5/3} - 40x^{2/3}$, so

$$f''(x) = \frac{40}{3}x^{2/3} - \frac{80}{3}x^{-1/3} = \frac{40}{3}x^{-1/3}(x-2).$$

$f''(x)$ is undefined when $x = 0$ and 0 when $x = 2$. Now, we can test points to make our sign chart. For example, $f''(-1) = \frac{40}{3}(-1)(-3) > 0$, $f''(1) = \frac{40}{3}(1)(-1) < 0$, and $f''(3) = \frac{40}{3}(3^{-1/3})(1) > 0$:

		0		2	
sign of f''	$+$		$-$		$+$

So, $\boxed{f \text{ is concave down on } (0,2) \text{ and concave up everywhere else}}$.

(f) *What are the inflection points of f?*

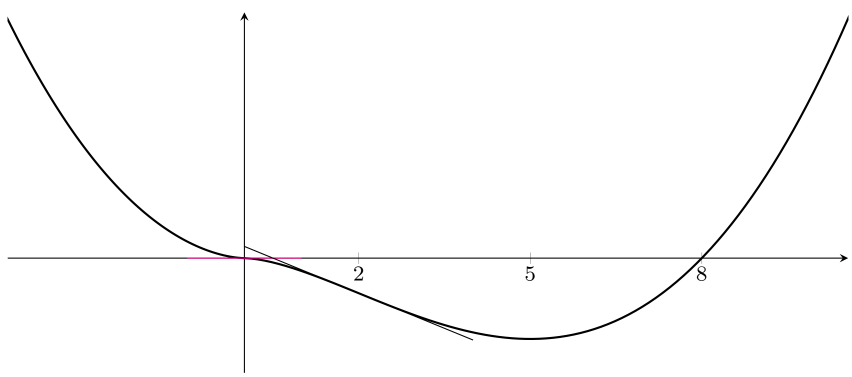
Solution. The concavity of $f(x)$ changes at $x = 0$ and $x = 2$, so there are two inflection points, $\boxed{(0,0) \text{ and } \left(2, -18 \cdot 2^{5/3}\right)}$.

(g) *Can you find the x- and y-intercepts of f?*

Solution. The y-intercept is $f(0) = 0$. The x-intercepts are where $f(x) = 0$, which happens when $x = 0$ or $x = 8$.

(h) *Sketch a rough graph of $f(x)$ that incorporates all of the above information.*

Solution.



(We've drawn the tangent lines at the inflection points to make the concavity change there clearer.)

28. *Graph* $f(x) = \dfrac{x}{\sqrt{4-x^2}}$. *Show how you found the domain, the intervals on which the function is increasing and decreasing, the intervals on which the function is concave up and concave down, the local maxima and minima, any asymptotes, and anything else of interest on the curve.*

Solution. Let's look at the key features.

- *Domain and discontinuities:* There are two things that could potentially make f undefined: taking the square root of a negative number or dividing by 0. The former happens if $x^2 > 4$, or when

$|x| > 2$. The latter happens when $4 = x^2$, or $|x| = 2$. So, the domain of f is $|x| < 2$, or $(-2, 2)$. On this domain, f is made up of familiar functions, so it has no discontinuities.

- *Symmetry:* Let's check whether f has useful symmetry: $f(-x) = \frac{-x}{\sqrt{4-(-x)^2}} = -\frac{x}{\sqrt{4-x^2}} = -f(x)$, so f is an odd function. Therefore, we can analyze it on just $[0, 2)$ and then use symmetry to graph it on $(-2, 0)$.

- *Intercepts*: f has its y-intercept at $f(0) = 0$. Its only x-intercept is at $x = 0$.

- *End behavior:* We should look at $\lim_{x \to 2^-} f(x)$ to understand the behavior of f at the right end of its domain. As $x \to 2^-$, $\sqrt{4 - x^2} \to 0^+$, so $\frac{x}{\sqrt{4-x^2}} \to \infty$.

- *Increasing / decreasing:* Let's make a sign chart for f'. First, we can rewrite $f(x) = x(4-x^2)^{-1/2}$; then,

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\boxed{x} \cdot \boxed{\left(4-x^2\right)^{-1/2}}\right) \\
&= x \cdot \frac{d}{dx}\left(\boxed{\boxed{\left(4-x^2\right)}^{-1/2}}\right) + \left(4-x^2\right)^{-1/2} \\
&= x \cdot \left(-\frac{1}{2}\left(4-x^2\right)^{-3/2}(-2x)\right) + \left(4-x^2\right)^{-1/2}
\end{aligned}
$$

Factoring out $(4-x^2)^{-3/2}$, we can rewrite this as:

$$
\begin{aligned}
&= \left(4-x^2\right)^{-3/2}\left(x^2+4-x^2\right) \\
&= \frac{4}{(4-x^2)^{3/2}}
\end{aligned}
$$

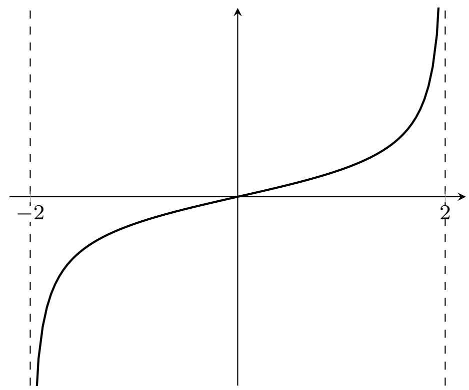
On $[0, 2)$, this is never undefined and never 0. So, the sign of f' does not change on $[0, 2)$. We can test any point to determine this sign; for example, $f'(1) = \frac{4}{3^{3/2}} > 0$, so f is increasing on $[0, 2)$.

- *Concavity:* Let's make a sign chart for f''. Since $f'(x) = 4(4-x^2)^{-3/2}$,

$$
\begin{aligned}
f''(x) &= 4 \cdot \frac{d}{dx}\left(\boxed{\boxed{\left(4-x^2\right)}^{-3/2}}\right) \\
&= -6\left(4-x^2\right)^{-5/2}(-2x) \\
&= \frac{12x}{(4-x^2)^{5/2}}
\end{aligned}
$$

This is never undefined on $[0, 2)$, and it's 0 only at $x = 0$. So, the sign of f'' does not change on $(0, 2)$. We can test any point to determine this sign; for example, $f''(1) = \frac{12}{3^{5/2}} > 0$. So, f is concave up on $(0, 2)$.

Putting this together, we get the graph of f on $[0, 2)$. Then we use the fact that f is odd to graph f on $(-2, 0)$:



29. Let $f(x) = \arcsin(4x^2)$.

(a) *What is the domain of $f(x)$?*

Solution. $\arcsin u$ is defined for u in $[-1, 1]$, which we can think of as $|u| \le 1$. So, $\arcsin(4x^2)$ is defined when

$$|4x^2| \le 1$$
$$|x^2| \le \frac{1}{4}$$

Square rooting both sides,

$$|x| \le \frac{1}{2}$$

So, the domain of $f(x)$ is $\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right]}$.

(b) *Is $f(x)$ an even function, an odd function, or neither?*

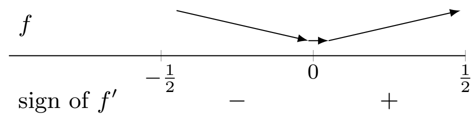
Solution. To determine this, we look at $f(-x)$, which is equal to $\arcsin\left[4(-x)^2\right] = \arcsin(4x^2) = f(x)$. So, f is $\boxed{\text{even}}$.

(c) *On what intervals is f increasing? On what intervals is it decreasing?*

Solution. To answer this, let's make a sign chart for f', since we know that the sign of f' tells us whether f is increasing or decreasing. First, we need to calculate f': $f'(x) = \frac{1}{\sqrt{1-(4x^2)^2}} \cdot 8x = \frac{8x}{\sqrt{1-16x^4}}$. Next, we need to figure out where this is 0 or undefined. This is undefined when the denominator is 0:

$$\sqrt{1-16x^4} = 0$$
$$1 - 16x^4 = 0$$
$$1 = 16x^4$$
$$\frac{1}{16} = x^4$$
$$\pm\frac{1}{2} = x$$

It's 0 when the numerator is 0, which happens at $x = 0$. Then, we can test points to get the following sign chart for f':



$\boxed{\text{So, } f \text{ is decreasing on } \left(-\frac{1}{2}, 0\right) \text{ and increasing on } \left(0, \frac{1}{2}\right).}$

(d) *Does f have an absolute maximum and absolute minimum on its domain? If so, find the absolute maximum and minimum values, and say where they occur.*

Solution. Since f is continuous and its domain is a closed interval, it must have an absolute maximum and absolute minimum by the Extreme Value Theorem. From the sign chart for f', we drew in (c), we can be sure that the absolute minimum occurs at $x = 0$, and the absolute minimum value is $f(0) = 0$.

From the sign chart for f', the absolute maximum could be at either endpoint of the domain; since we found that f is even, the function must be equal at $x = \pm\frac{1}{2}$. So, the absolute maximum value is $f\left(\pm\frac{1}{2}\right) = \arcsin(1) = \frac{\pi}{2}$. Thus:

The absolute minimum point is $(0, 0)$, and the absolute maximum points are $\left(-\frac{1}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

(e) *In (a), you should have found that the domain of f was a closed interval $[a, b]$. What are $\lim_{x \to a^+} f'(x)$ and $\lim_{x \to b^-} f'(x)$? (Note that this is asking about f', not about f.) What does this tell you about the graph of f?*

Solution. We are asked to calculate $\displaystyle\lim_{x \to (-1/2)^+} \frac{8x}{\sqrt{1 - 16x^4}}$ and $\displaystyle\lim_{x \to (1/2)^-} \frac{8x}{\sqrt{1 - 16x^4}}$.

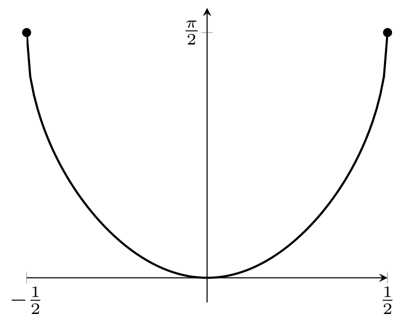
As $x \to \left(-\frac{1}{2}\right)^+$, $8x \to -4$ and $\sqrt{1 - 16x^4} \to 0$; when we're looking at the limit of a fraction where the numerator has a non-zero limit and the denominator $\to 0$, we need to think about the sign of the denominator. Here, the denominator $\sqrt{1 - 16x^4}$ is always positive, so $\boxed{\displaystyle\lim_{x \to (-1/2)^+} f'(x) = -\infty}$.

(When x is just a bit bigger than $-\frac{1}{2}$, evaluating $\dfrac{8x}{\sqrt{1 - 16x^4}}$ involves dividing a number close to -4 by a tiny positive number, which results in a very negative answer.)

As $x \to \left(\frac{1}{2}\right)^-$, $8x \to 4$ and $\sqrt{1 - 16x^4} \to 0$. Since $\sqrt{1 - 16x^4}$ is always positive, $\boxed{\displaystyle\lim_{x \to (1/2)^-} f'(x) = \infty}$.

(f) *Use the above information to sketch a rough graph of f. (Your sketch need not accurately reflect the concavity of the graph.)*

Solution. Here is the graph of $f(x)$:



✓ When you sketch a graph, the key is to be consistent with everything else you've found. Here, that means that your graph should:

- Have domain $\left[-\frac{1}{2}, \frac{1}{2}\right]$.
- Be symmetric over the y-axis.
- Decrease on $\left(-\frac{1}{2}, 0\right)$ and increase on $\left(0, \frac{1}{2}\right)$.
- Be vertical at $x = \pm\frac{1}{2}$ (because of the limits in (e).
- Have its highest points at $\left(-\frac{1}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ and its lowest point at $(0, 0)$.

30. *Determine how many roots* $f(x) = x^3 + x - 2$ *has, if any, on the interval* x *in* [0,3].

    Solution. We've tackled something similar in spirit in the past. Revisit the handout and homework on the Intermediate Value Theorem!

    But essentially, before we start counting, let's think about whether there is a root at all on this interval.

    Note that the function f is continuous on [0,3] and differentiable on (0,3). Since $f(0) = -2$ and $f(3) = 28$, by the intermediate value theorem, we know that the function takes all values between -2 and 28, and so specifically, it must hit the value 0.

    So there is at least one root between [0,3]. Are there more? Suppose there were two points a,b where $f(a) = f(b) = 0$. Then, since this function is continuous and differentiable everywhere, by the Mean Value Theorem, there must be a point c in (a,b) such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

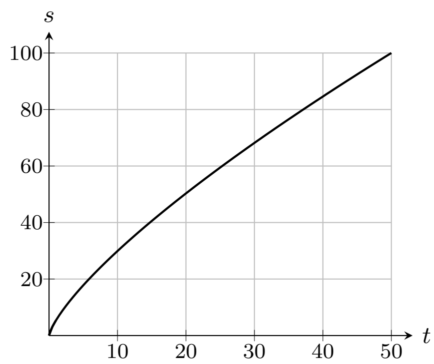
    But $\frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$. So we are saying that there must be a point where $f'(c) = 0$. But $f'(x) = 3x^2 + 1$ and this is never equal to zero. Hence, f must have exactly one root in the interval [0,3].

31. Show that $f(x) = x + \ln(x)$ has a zero.

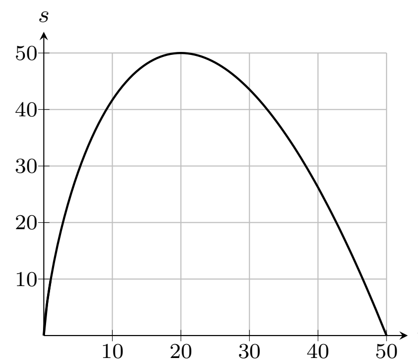
    Solution. $\lim_{x\to 0^+} f(x) = -\infty$, and $f(1) = 1 + 0 = 1$. By the IVT, f takes all values in $(-\infty$,1), including 0.

32. *Average speed vs. average velocity.* A swimmer is swimming a 100 m long race, which is one lap in a 50 m long pool. Let $s(t)$ be his distance from the starting position t seconds after the start of the race.

    (a) *Which of the following is a more reasonable graph for* $s(t)$*? Why?*



(A)



(B)

Solution. Choice (B) is the reasonable one: when the swimmer reaches the opposite end of the pool, he is 50 m from the starting point. At the end of the race, he is back at the starting point, so he is 0 m from the starting point. (Choice (A) would say that, at the end of the race, the swimmer is 100 m from his starting point.)

(b) *According to the graph you chose, what was the swimmer's average speed for the race? Average velocity for the race?*

Solution. First, remember that average speed $= \frac{\text{change in distance traveled}}{\text{change in time}}$, while average velocity $= \frac{\text{change in position}}{\text{change in time}}$.

When the swimmer starts the race, he has traveled 0 m; when the swimmer ends, he has traveled 100 m; therefore, the change in his distance traveled is 100 m. The race takes 50 seconds, so his average speed for the race is $\frac{100\text{ m}}{50\text{ s}} = 2$ m/s.

The swimmer's change in position for the race is 0 because he ends in the same place he started, so his average velocity for the race is $\frac{0\text{ m}}{50\text{ s}} = 0$ m/s.

(c) *What was the swimmer's average speed over the first 20 seconds of the race? Average velocity?*

Solution. In the first 20 seconds, the swimmer swam a distance of 50 m. So, his average speed on this interval was $\frac{50\text{ m}}{20\text{ s}} = 2.5$ m/s.

The swimmer's average velocity over the first 20 seconds is $\frac{\text{change in position}}{\text{change in time}} = \frac{s(20)-s(0)}{20-0} = \frac{50\text{ m}}{20\text{ s}} =$ 2.5 m/s.

(d) *What was the swimmer's average speed over the last 50 m of the race? Average velocity?*

Solution. The swimmer swam the last 50 m in 30 seconds, so his average speed was $\frac{50\text{ m}}{30\text{ s}} = \frac{5}{3}$ m/s.

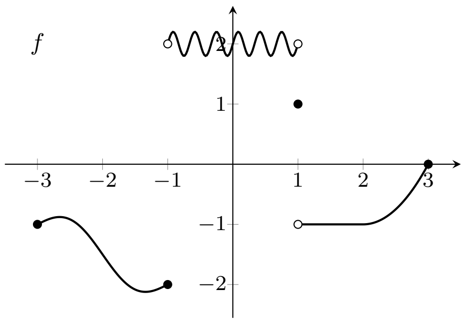
His average velocity on this interval was $\frac{s(50)-s(20)}{50-20} = \frac{-50\text{ m}}{30\text{ s}} = -\frac{5}{3}$ m/s.

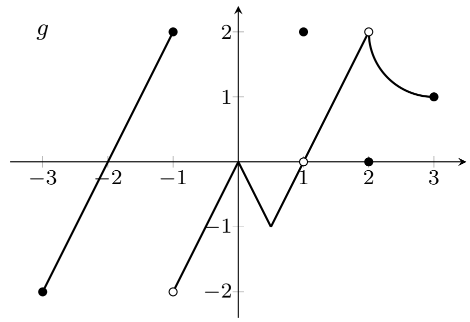
(e) *What is the difference between velocity and speed?*

Solution. As we said already, average velocity over an interval is $\frac{\text{change in position}}{\text{change in time}}$, while average speed is $\frac{\text{change in distance traveled}}{\text{change in time}}$. Just knowing the value of one of these quantities does not tell us the other; for example, in (b), knowing that the average speed for the race is 2 m/s doesn't tell us what the average velocity is, and knowing that the average velocity is 0 m/s doesn't tell us the swimmer's average speed.

On the other hand, instantaneous speed is simply the absolute value of instantaneous velocity.

33. The graphs of f and g are given below.





Evaluate each of the following limits.

(a) $\lim_{x\to 2}(g(x) - f(x))$

Solution. As $x \to 2$, $f(x) \to -1$ and $g(x) \to 2$, so $\lim_{x\to 2}(g(x) - f(x)) = \lim_{x\to 2}(2+1) = \boxed{3}$.

(b) $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$

Solution. As $x \to 0$, $f(x) \to 2$ and $g(x) \to 0$. In this situation (where the denominator tends to 0 but the numerator doesn't), we need to think about the sign of the denominator. Here, $g(x)$ is negative for x near 0, so $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \boxed{-\infty}$. (After all, for x very close to 0, $f(x)$ is very close to 2, and $g(x)$ is a very slightly negative number. When you divide a number close to 2 by a slightly negative number, you get a very negative result.)

(c) $\lim_{x\to -2} \dfrac{f(x)}{g(x)}$

Solution. As $x \to -2$, $f(x)$ approaches $f(-2)$, which is a negative number (although we can't tell its exact value easily from the graph), while $g(x)$ approaches 0. So, we should again think about the sign of the denominator. Here, the sign changes at $x = -2$. For x just a bit less than -2, $g(x)$ is negative, so $\lim_{x\to -2^-} \dfrac{f(x)}{g(x)} = \infty$. For x just a bit bigger than -2, $g(x)$ is positive, so $\lim_{x\to 2^+} \dfrac{f(x)}{g(x)} = -\infty$. Therefore, $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$ $\boxed{\text{does not exist}}$.

(d) $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$

Solution. As $x \to 1$, $f(x)$ does not approach a limit because the one-sided limits don't agree. Therefore, we should look at the given limit from both sides. As $x \to 1^-$, $f(x) \to 2$ and $g(x) \to 0^-$,[2] so $\frac{f(x)}{g(x)} \to -\infty$. As $x \to 1^+$, $f(x) \to -1$ and $g(x) \to 0^+$, so $\frac{f(x)}{g(x)} \to -\infty$. Since $\lim_{x\to 1^-} \dfrac{f(x)}{g(x)}$ and $\lim_{x\to 1^+} \dfrac{f(x)}{g(x)}$ are both $-\infty$, $\lim_{x\to 1} \dfrac{f(x)}{g(x)} = \boxed{-\infty}$.

[2] Remember that writing, "As $x \to 1^-$, $g(x) \to 0^-$" is shorthand for "As $x \to 1^-$, $g(x)$ approaches 0 from below." That is, when x is a bit less than 1, $g(x)$ is a bit less than 0.

(e) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$

Solution. As $x \to 2$, $f(x) \to -1$ and $g(x) \to 2$, so $\dfrac{f(x)}{g(x)} \to \boxed{-\dfrac{1}{2}}$.

34. *Working algebraically with limits.* Can you evaluate the following limits?

(a) $\lim_{x\to \frac{\pi}{3}} \dfrac{\sin x}{\cos x}$

Solution. $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. So:

$$\begin{aligned} \lim_{x\to \frac{\pi}{3}} \frac{\sin x}{\cos x} &= \lim_{x\to \frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= \lim_{x\to \frac{\pi}{3}} \sqrt{3} \\ &= \boxed{\sqrt{3}} \end{aligned}$$

(b) $\lim_{x\to 3\pi^+} \dfrac{\sin x}{\cos x}$

Solution. $\sin(3\pi) = 0$ and $\cos(3\pi) = -1$. So:

$$\begin{aligned} \lim_{x\to 3\pi^+} \frac{\sin x}{\cos x} &= \lim_{x\to 3\pi^+} \frac{0}{-1} \\ &= \lim_{x\to 3\pi^+} 0 \\ &= \boxed{0} \end{aligned}$$

(c) $\lim_{x\to 3\pi^+} \dfrac{\cos x}{\sin x}$

Solution. $\sin(3\pi) = 0$ and $\cos(3\pi) = -1$. So when we try to simply "plug in the value", we get a limit of the form $\left(\frac{-1}{0}\right)$. So the limit $\boxed{\text{does not exist}}$. However, let's see if we can say more.

In this situation (where the denominator tends to 0 but the numerator doesn't), we need to think about the sign of the denominator. When x is just a little more than 3π, then looking at the graph of $\sin x$, we can see that $\sin x$ is negative, so $\lim_{x\to 3\pi^+} \sin(x) = 0^-$ (that is, as $x \to 3\pi^+$, $\sin x$ approaches 0 from the negative side); When we take -1 and divide it by a tiny negative number, we get a huge positive number. Therefore, $\lim_{x\to 3\pi^+} \dfrac{\cos x}{\sin x} = \boxed{\infty}$.

(d) $\lim_{x\to 3\pi} \dfrac{\cos x}{\sin x}$

Solution. Again this is a limit where the denominator tends to 0 but the numerator doesn't go to 0. We have seen already that $\lim_{x\to 3\pi^+} \dfrac{\cos x}{\sin x} = \infty$. Let's compute $\lim_{x\to 3\pi^-} \dfrac{\cos x}{\sin x}$. When x is just a little less than 3π, then looking at the graph of $\sin x$, we can see that $\sin x$ is positive, so $\lim_{x\to 3\pi^-} \sin(x) = 0^+$ (that is, as $x \to 3\pi^-$, $\sin x$ approaches 0 from the positive side); When we take -1 and divide it by a tiny positive number, we get a huge negative number. Therefore, $\lim_{x\to 3\pi^-} \dfrac{\cos x}{\sin x} = -\infty$. Since the two limits are different, the limit $\lim_{x\to 3\pi} \dfrac{\cos x}{\sin x}$ $\boxed{\text{does not exist}}$.

(e) $\lim_{x\to 2} \frac{(\sin x) - 2}{(x-2)^2}$

Solution. As x approaches 2, the numerator here approaches $\sin 2 - 2$, while the denominator approaches 0. Note that since $-1 \le \sin x \le 1$, we know that $\sin 2 - 2 < 0$. So this is a limit where the denominator tends to 0 but the numerator doesn't go to 0. Let us consider the sign of the denominator. Since the denominator is a square, it will always be positive. So $\lim_{x\to 2}(x-2)^2 = 0^+$ (that is, when x approaches 0 from either side, $(x-2)^2$ approaches 0 from the positive side).

Since a negative number divided by a tiny positive number is a huge negative number, we get that $\lim_{x\to 2} \frac{(\sin x) - 2}{(x-2)^2} = \boxed{-\infty}$.

(f) $\lim_{x\to 3^+} \ln(x^2 - 9)$

Solution. Let's start by thinking about how the limit is affecting the x values here. As x approaches 3, (x^2-9) approaches 0. However, here, we are actually approaching 3 from the right. When x is just a bit bigger than 3, then $x^2 - 9$ is just a bit bigger than 0. What is ln of a tiny positive number? Looking at the graph of ln, we can see that ln of a tiny positive number is a really large negative number. So $\lim_{x\to 3^+} \ln(x^2-9) = \boxed{-\infty}$.

35. In this problem, we'll look at $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$.

(a) *Aberforth is thinking about this limit and says, "As $x \to 0$, x^2 approaches 0. 0 times anything is 0, so $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$ must be 0." What do you think of Aberforth's reasoning?*

Solution. Aberforth's reasoning is incorrect, which we can see from an example like $\lim_{x\to 0} x^2 \cdot \frac{1}{x^4}$. Aberforth would think that this limit is 0, but we can use algebra to rewrite it as $\lim_{x\to 0} \frac{1}{x^2}$, which is ∞.

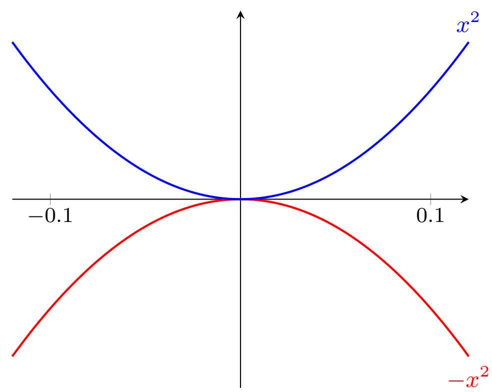
In general, it's never safe to look at just one piece of a limit; you always need to look at all pieces of a limit to understand it!

(b) *Find $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$. Explain your reasoning carefully.*

Solution. Aberforth's mistake was ignoring the $\sin\left(\frac{\pi}{x}\right)$ piece of this limit, so let's try to understand that piece. We've seen in **??** that it doesn't approach a limit as $x \to 0$ and that it actually behaves very wildly near $x = 0$. However, one simple thing we can say about it is that $\sin\left(\frac{\pi}{x}\right)$ is always between -1 and 1, just because sine of *anything* is between -1 and 1. Therefore, $x^2 \sin\left(\frac{\pi}{x}\right)$ is always between $x^2(-1) = -x^2$ and $x^2(1) = x^2$. That is

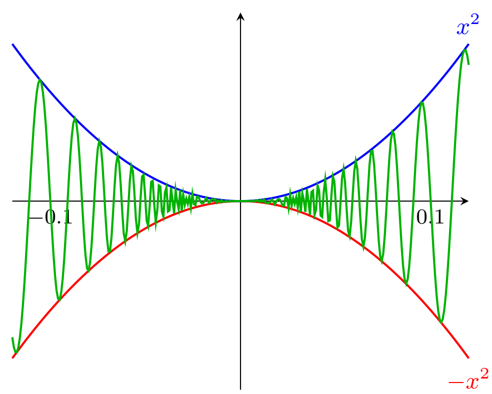
$$-x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le x^2$$

But if we graph $-x^2$ and x^2, we see that any function that is stuck between them must tend to 0 as $x \to 0$, because both x^2 and $-x^2$ tend to 0 as $x \to 0$:



(c) *Sketch a rough graph of* $f(x) = x^2 \sin\left(\frac{\pi}{x}\right)$ *for* x *near 0.*

Solution. As we saw in **??**, around $x = 0$, $\sin\left(\frac{\pi}{x}\right)$ oscillates (wildly!) between -1 and 1. Therefore, $x^2 \sin\left(\frac{\pi}{x}\right)$ oscillates between $x^2(-1) = -x^2$ and $x^2(1) = x^2$. Its graph is the green one below:



36. Evaluate the limits below. If the limit does not exists, write whether it is ∞ or $-\infty$. If it does not exists and also not $\pm\infty$, explain why the limit does not exist

(a) $\lim_{x\to 0} \frac{3x - x\cos x}{\sin 2x}$.

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{3x - x\cos x}{\sin 2x} &= \lim_{x\to 0} \frac{3x}{\sin 2x} - \lim_{x\to 0} \frac{x}{\sin 2x} \cdot \cos x \\
&= \lim_{x\to 0} \frac{2x}{\sin 2x} \cdot \frac{3}{2} - \lim_{x\to 0} \frac{2x}{\sin 2x} \cdot \frac{\cos x}{2} \\
&= 1 \cdot \frac{3}{2} - 1 \cdot \frac{1}{2} \\
&= 1.
\end{aligned}$$

(b) $\lim_{x\to 2} \frac{|x-2|}{x-2}$.

Solution. $|x-2|$ is a piecewise function

$$|x-2| = \begin{cases} x-2, & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$$

Evaluate the left and right limit separately

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1.$$

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1.$$

Left limit does not equal to the right limit. The limit does not exist, and also not $\pm\infty$.

(c) $\lim\limits_{x\to\infty} 2x - \sqrt{4x^2+1}$.

Solution.

$$\begin{aligned}
\lim_{x\to\infty} 2x - \sqrt{4x^2+1} &= \lim_{x\to\infty} \frac{(2x-\sqrt{4x^2+1})(2x+\sqrt{4x^2+1})}{2x+\sqrt{4x^2+1}} \\
&= \lim_{x\to\infty} \frac{4x^2-(4x^2+1)}{2x+\sqrt{4x^2+1}} \\
&= \lim_{x\to\infty} \frac{-1}{2x+\sqrt{4x^2+1}} \\
&= 0
\end{aligned}$$

(d) $\lim\limits_{x\to 0} \csc(2x)\tan(3x)$

Solution.

$$\begin{aligned}
\lim_{x\to 0} \csc(2x)\tan(3x) &= \lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)\cos(3x)} \\
&= \lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} \cdot \frac{1}{\cos(3x)} \\
&= \lim_{x\to 0} \frac{3}{2} \cdot \frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{1}{\cos(3x)} \\
&= \frac{3}{2} \cdot 1 \cdot 1 \cdot \frac{1}{1} \\
&= \frac{3}{2}
\end{aligned}$$

(e) $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-3x+2}$

Solution.

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{x+2}{x-2} = \frac{3}{-1} = -3$$

(f) $\lim_{x\to\infty} \frac{\sqrt{2x^2+2x+3}}{3x-1}$

Solution.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+2x+3}}{3x-1} = \lim_{x\to\infty} \frac{\sqrt{2x^2+2x+3}}{3x-1} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{\sqrt{2+2/x+3/x^2}}{3-1/x} = \frac{\sqrt{2}}{3}$$

(g) $\lim_{x\to\infty} \frac{e^{2x}+e^x+4}{7e^{2x}+e^x-4}$

Solution. First, as $x \to \infty$, $e^{2x}+e^x+4 \to \infty$. As $x \to \infty$, the denominator also $\to \infty$. We should read an $\frac{\infty}{\infty}$ limit as a "needs more work" limit! We'll follow our technique of dividing the numerator and denominator by the largest term in the denominator. Note that e^{2x} is the largest term in the denominator, so we'll rewrite

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{2x}+e^x+4}{7e^{2x}+e^x-4} &= \lim_{x\to\infty} \frac{\left(e^{2x}+e^x+4\right)/e^{2x}}{\left(e^{2x}+e^x-4\right)/e^{2x}} \\ &= \lim_{x\to\infty} \frac{\frac{e^{2x}}{e^{2x}}+\frac{e^x}{e^{2x}}+\frac{4}{e^{2x}}}{\frac{7e^{2x}}{e^{2x}}+\frac{e^x}{e^{2x}}-\frac{4}{e^{2x}}} \\ &= \lim_{x\to\infty} \frac{1+\frac{1}{e^x}+\frac{4}{e^{2x}}}{7+\frac{1}{e^x}-\frac{4}{e^{2x}}}\end{aligned}$$

As $x \to \infty$, $\frac{1}{e^x} \to 0$ and $\frac{4}{e^{2x}} \to 0$, so the numerator $\to 1$ while the denominator $\to 7$:

$$= \boxed{\frac{1}{7}}$$

(h) $\lim_{x\to-\infty} \frac{e^{2x}+e^x+4}{7e^{2x}+e^x-4}$

Solution. You might see this, and immediately jump to our strategy of dividing the numerator and denominator by the largest term in the denominator. However, that wouldn't work here! That's a strategy that only works with limits of the form $\left(\frac{\infty}{\infty}\right)$. Here, $x \to -\infty$, $e^x \to 0$, and $e^{2x} \to 0$. So the numerator $\to 4$, and the denominator $\to -4$. That is

$$\lim_{x\to-\infty} \frac{e^{2x}+e^x+4}{7e^{2x}+e^x-4} = \frac{4}{-4} = \boxed{-1}$$

.

(i) $\lim_{x\to\infty} \frac{70^{x/2}-1{,}000{,}000}{2^{3x}+3^{2x}+10^{100}}$

Solution. First, as $x \to \infty$, the denominator and denominator, both go $\to \infty$. So we'll use our technique of dividing the numerator and denominator by the largest term in the denominator. But what is the largest term in the denominator? It is easier to compare terms when the exponents are the same. Note that we can rewrite $2^{3x} = 8^x$ and $3^{2x} = 9^x$, and now it is clear that 3^{2x} or 9^x is the largest term. We will divide both the numerator and denominator by 9^x. Note here that we can rewrite $\dfrac{70^{x/2}}{9^x} = \left(\dfrac{\sqrt{70}}{9}\right)^x$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{70^{x/2} - 1{,}000{,}000}{2^{3x} + 3^{2x} + 10^{100}} &= \lim_{x\to\infty} \frac{(\sqrt{70})^x - 1{,}000{,}000}{8^x + 9^x + 10^{100}} \\
&= \lim_{x\to\infty} \frac{(\sqrt{70}^x - 1{,}000{,}000)/9^x}{(8^x + 9^x + 10^{100})/9^x} \\
&= \lim_{x\to\infty} \frac{\left(\frac{\sqrt{70}}{9}\right)^x - \frac{1{,}000{,}000}{9^x}}{\left(\frac{8}{9}\right)^x + 1 + \frac{10^{100}}{9^x}}
\end{aligned}$$

Since $\frac{\sqrt{70}}{9} < 1$, as $x \to \infty$, $\left(\frac{\sqrt{70}}{9}\right)^x \to 0$. Similarly $\left(\frac{8}{9}\right)^x \to 0$, $\frac{10^{100}}{9^x} \to 0$, and $\frac{1{,}000{,}000}{9^x} \to 0$. So the numerator $\to 0$ while the denominator $\to 1$:

$$\begin{aligned}
&= \frac{0}{1} \\
&= \boxed{0}
\end{aligned}$$

(j) $\displaystyle\lim_{x\to-\infty} \sin\left(\frac{1}{x}\right)$

Solution. Note that as $x \to -\infty$, we have that $\frac{1}{x} \to 0$. Since sine of a number really close to 0, is a number really close to 0, we have that $\displaystyle\lim_{x\to-\infty} \sin\left(\frac{1}{x}\right) = \boxed{0}$.

37. *Let a be a constant and f be the function defined by*

$$f(x) = \begin{cases} \frac{ax}{6} & \text{for } x < 1 \\ \frac{1}{x+a} & \text{for } x \geq 1 \end{cases}.$$

Find all values of a for which f is continuous on $(-\infty, \infty)$. (Do this without using a calculator!)

Solution. No matter what a is, $\frac{ax}{6}$ is continuous for $x < 1$ (it's linear). However, $\frac{1}{x+a}$ may not be continuous for $x \geq 1$: it has a vertical asymptote at $x = -a$, so $\frac{1}{x+a}$ will have a discontinuity on $x \geq 1$ if $-a \geq 1$ (i.e., if $-1 \geq a$). So, the first thing we know is that we need $-1 < a$.

Now, we also need to figure out what would make f continuous at $x = 1$. We want $\lim_{x\to 1} f(x) = f(1) = \dfrac{1}{1+a}$, so we need both $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$ to equal $\frac{1}{1+a}$. That is, we want $\displaystyle\lim_{x\to 1^-} \frac{ax}{6} = \frac{1}{1+a}$ and $\displaystyle\lim_{x\to 1^+} \frac{1}{x+a} = \frac{1}{1+a}$. The latter is true for all a, but the former is only true if

$$\begin{aligned}
\frac{a}{6} &= \frac{1}{1+a} \\
a(1+a) &= 6 \\
a^2 + a - 6 &= 0 \\
(a+3)(a-2) &= 0 \\
a &= -3, 2
\end{aligned}$$

Since we also need $a > -1$, the only value of a that works is $\boxed{a = 2}$.

38. Let f be the function defined by

$$f(x) = \begin{cases} -x^2 + 1 & \text{for } x \leq 1 \\ ax + b & \text{for } x > 1 \end{cases}$$

(a) *For what values of a and b will f be continuous at $x = 1$?*

Solution. If we want the function to be continuous at $x = 1$, we need to make sure that

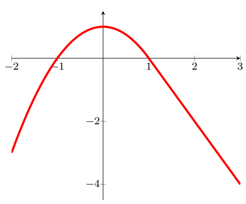
$$\lim_{x \to 1} f(x) = f(1)$$

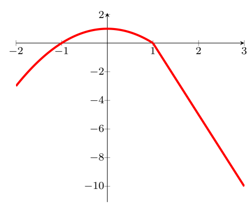
For the limit to exist, the left hand limit needs to equal the right hand limit.

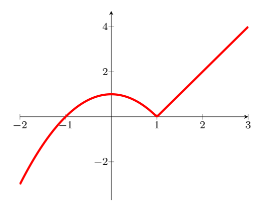
$$\lim_{x \to 1^+} f(x) = a + b$$

$$\lim_{x \to 1^-} f(x) = 0$$

Therefore, as long as $a = -b$ the function is continuous. Lets draw a few examples:







Note that based off the pictures, it makes sense that there are infinitely many solutions to consider here!

(b) *Are there any values of a and b for which f will be differentiable at $x = 1$?*

Solution. We want to avoid having a sharp corner at $x = 1$. To accomplish this we want the derivatives of the two pieces to match up.

$$\frac{d}{dx}(-x^2 + 1) = -2x$$

$$\frac{d}{dx}(ax + b) = a$$

At $x = 1$ we need $-2x = a$, so $a = -2$. This graph is pictured on the left above. Again, thinking graphically, it makes sense why now we should expect a unique solution.

39. For each of the following functions, do the following:

- Use the **limit definition of the derivative** to compute the slope at the given point. No credit will be given for using any other method.
- Use your answer for the slope to find the equation of the tangent line at the given point.

(a) $f(x) = \sin x$ *at* $x = 0$.

Solution. By the definition of the derivative,

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \\
&= \boxed{1}
\end{aligned}
$$

Slope of the tangent line at $x = 0$ is $f'(0) = 1$. Since, the tangent line is passing through the point $(0,f(0)) = (0,0)$, the equation of the tangent line at $x = 0$ is $\boxed{y = x}$.

(b) $f(x) = \dfrac{1-x}{x}$ *at* $x = 1$.

Solution. By the definition of the derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1-x-h}{x+h} - \frac{1-x}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - 1 - \frac{1}{x} + 1}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= -\frac{1}{x^2}
\end{aligned}
$$

Slope of the tangent line is $f'(1) = \boxed{-1}$. The tangent line passing through the point $(1,f(1)) = (1,0)$. So the equation of the tangent line at $x = 1$ is $\boxed{y = -(x-1)}$.

40. Consider the function $f(x) = \dfrac{1}{\sqrt{x+1}}$

(a) *Find $f'(x)$ using the formal definition of the derivative.*

Solution. By the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{(x+h)+1}} - \frac{1}{\sqrt{x+1}}}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{\sqrt{(x+h)+1}} - \frac{1}{\sqrt{x+1}} \right) \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{h\sqrt{(x+h)+1}} - \frac{1}{h\sqrt{x+1}} \right) \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x+1}}{h\sqrt{x+h+1}\sqrt{x+1}} - \frac{\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \right) \\
&= \lim_{h\to 0} \frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \cdot \frac{\sqrt{x+1}+\sqrt{x+h+1}}{\sqrt{x+1}+\sqrt{x+h+1}} \\
&= \lim_{h\to 0} \frac{(x+1)-(x+h+1)}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)} \\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)} \\
&= \frac{-1}{\sqrt{x+0+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+0+1}\right)} \\
&= \boxed{-\frac{1}{2(x+1)^{3/2}}}
\end{aligned}$$

(b) *Find x so that the tangent line to f at x has slope $-\frac{1}{16}$.*

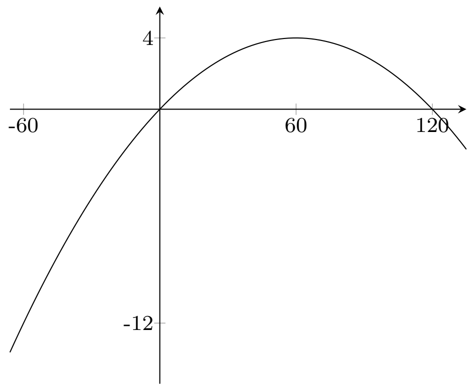
Solution. The slope of the tangent line is $f'(x)$, which we just found. So we just need to know when $f'(x) = -\frac{1}{16}$.

$$\begin{aligned}
-\frac{1}{2(x+1)^{3/2}} &= -\frac{1}{16} \\
2(x+1)^{3/2} &= 16
\end{aligned}$$

raising both sides to the $\frac{2}{3}$, we get:

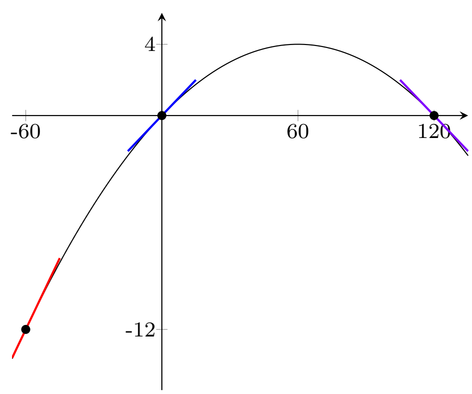
$$\begin{aligned}
(x+1)^{3/2} &= 8 \\
x+1 &= 8^{2/3} \\
x+1 &= \left(\sqrt[3]{8}\right)^2 \\
x+1 &= 4 \\
x &= \boxed{3}
\end{aligned}$$

41. One day, a hitchhiker is wandering along a highway near a gas station. Suppose his position at time t is $f(t) = 360t - 3t^2$, where t is measured in minutes after noon and the position is given in feet east of the gas station. Here is the graph of $f(t)$:



(a) *On the graph above, sketch slopes that represent the hitchhiker's instantaneous velocity at 11 am, 12:30 pm, and 2 pm. Which of these velocities is the biggest?*

Solution. The instantaneous velocity is given by the slope of the tangent line. The instantaneous velocity at 11 am ($t = -60$) is the slope of the red line. The instantaneous velocity at noon ($t = 0$) is the slope of the blue line. The instantaneous velocity at 2 pm ($t = 120$) is the slope of the purple line. The instantaneous velocity is biggest when the absolute value of slope is steepest. This happens at $t = -60$ or at 11 am. Note that as you go along a concave down graph, the slope decreases from left to right.



(b) *Were there any times when the hitchhiker's instantaneous velocity was 0? If so, when?*

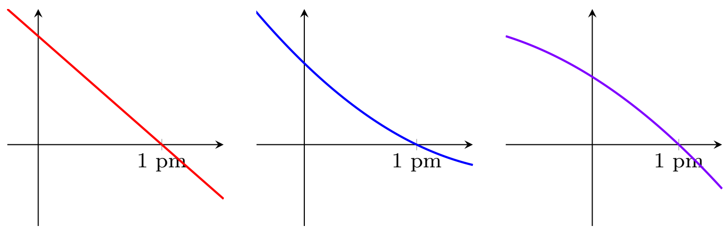
Solution. Yes! The slope is zero at $t = 60$ (or 1 pm) as the tangent line is horizontal at that point. So the instantaneous velocity is 0 at 1 pm.

(c) *At what times was the hitchhiker's instantaneous velocity positive?*

Solution. The instantaneous velocity is positive when the slope is positive. This happens at all times before 1 pm.

(d) *Sketch a rough graph of the hitchhiker's velocity as a function of time.*

Solution. Note that from our work above we see that before 1 pm, the velocity is positive (because $f(t)$ is increasing, so the slope is positive) and decreasing (because $f(t)$ is concave down so the slope is decreasing). After 1 pm the velocity is negative (because $f(t)$ is decreasing, so the slope is negative) and decreasing (because $f(t)$ is concave down so the slope is decreasing). At 1 pm, it is 0. Here are three possible sketches of the velocity:



(A) (B) (C)

So we see that the velocity, which was the derivative of the position function, is a function in its own right!

(e) *Using the definition of the derivative, calculate the hitchhiker's velocity at time t. This is denoted by $f'(t)$ or $\frac{df}{dt}$. Does this agree with your sketch?*

Solution. The instantaneous velocity is the derivative of the position function. This is denoted by $f'(t)$ or $\frac{df}{dt}$. We have that

$$\frac{df}{dt} = f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

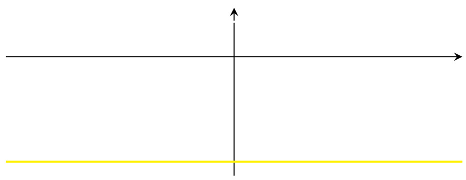
Let's compute this:

$$\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h\to 0} \frac{360(t+h) - 3(t+h)^2 - (360t - 3t^2)}{h} \\
&= \lim_{h\to 0} \frac{360(t+h) - 3(t+h)^2 - 360t + 3t^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{360t} + 360h - \cancel{360t} + 3t^2 - 3(t+h)^2}{h} \\
&= \lim_{h\to 0} \frac{360h + 3(t^2 - (t+h)^2)}{h} \\
&= \lim_{h\to 0} \frac{360h + 3((-h)(2t+h))}{h} \\
&= \lim_{h\to 0} \frac{360h - 3h(2t+h)}{h} \\
&= \lim_{h\to 0} (360 - 3(2t+h)) \\
&= 360 - 6t
\end{aligned}$$

So $v(t) = \frac{df}{dt} = f'(t) = 360 - 6t$. Ah-ha! Now we see that indeed this is a line with negative slope (hence decreasing), and so graph (A) in red is the correct velocity graph!

(f) *What is the hitchhiker's acceleration at time t? Sketch a graph of his acceleration as a function of time.*

Solution. Acceleration is the derivative of velocity. That is $a(t) = v'(t)$. Since $v(t)$ is a straight line, we know exactly what its slope is! The slope of the line $v(t) = 360 - 6t$ is -6. And so $a(t) = -6$. Here is the graph:



42. Sara, the owner of a cupcake truck, has been experimenting with the price of the cupcakes she sells. Unsurprisingly, she has found that the number of cupcakes she sells per day depends on the price. Let $C(p)$ be the number of cupcakes she sells when the price of a cupcake is p cents.

So far, Sara has found that, when the price of a cupcake is 300 cents, she sells 600 cupcakes a day.

(a) *What are the units of $C'(300)$?*

Solution. Since C is in cupcakes and p is in cents, $C'(300)$ is in $\boxed{\text{cupcakes per cent}}$. One way to understand this is by going back to the definition of the derivative: $C'(300) = \lim_{h\to 0} \frac{C(300+h) - C(300)}{h}$; the numerator $C(300+h) - C(300)$ is in cupcakes, and the denominator h is in cents. Alternatively, you could think about how we represent $C'(300)$ graphically: when we graph $C(p)$ (with units of cents on the horizontal axis and cupcakes on the vertical axis), it's a slope, so it must have units of cupcakes per cent.

(b) *Do you expect $C'(300)$ to be positive or negative? Why?*

Solution. We expect Sara to sell fewer cupcakes when the price is higher, so we expect $C(p)$ to be a decreasing function. Therefore, $C'(300)$ should be $\boxed{\text{negative}}$. (Tangent lines for a decreasing function will have negative slope.)

(c) *Suppose $C'(300) = -5$. If Sara raises the price of cupcakes to 310 cents, how many cupcakes do you expect her to sell per day?*

Solution. Saying $C'(300) = -5$ means that, when the price of cupcakes is 300 cents, the instantaneous rate of change in the number of cupcakes Sara sells with respect to price is -5 cupcakes per cent. In other words, for each cent Sara raises the price above 300, we expect her to sell 5 fewer cupcakes. Therefore, when she raises the price from 300 to 310 (an increase of 10 cents), we expect her to sell 50 fewer cupcakes, so we expect her to sell about $600 - 50 = \boxed{550}$ cupcakes.

43. *Let $f(x) = \left|\frac{x^4 - 9x^2 + 20}{x^2 - 4}\right|$. Sketch the graph of f, and then sketch the graph of f'.*

Solution. The formula for f looks complicated, but we can simplify it:

$$\left|\frac{x^4 - 9x^2 + 20}{x^2 - 4}\right| = \left|\frac{(x^2 - 4)(x^2 - 5)}{x^2 - 4}\right|$$

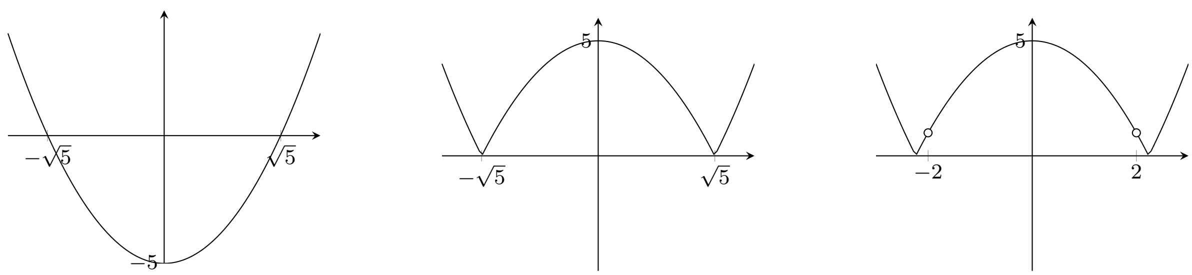
As long as $x^2 - 4 \neq 0$, we can cancel the $x^2 - 4$ in the numerator and denominator:

$$= |x^2 - 5| \text{ if } x^2 - 4 \neq 0$$

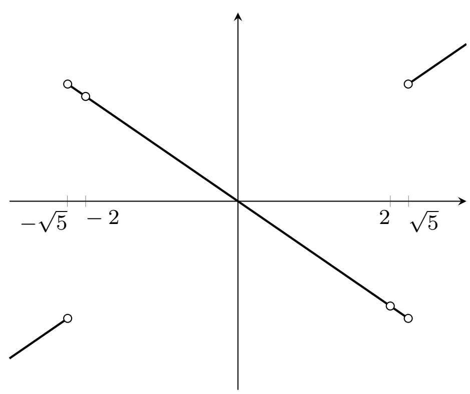
When $x^2 - 4 = 0$, $f(x)$ is undefined. So,

$$f(x) = \begin{cases} |x^2 - 5| & \text{if } x \neq \pm 2 \\ \text{undefined} & \text{if } x = \pm 2 \end{cases}$$

The left picture shows the graph of $x^2 - 5$, the middle picture is the graph of $|x^2 - 5|$, and the right picture is the graph of f:



The derivative of f looks like this:



Notes:

- From the graph of f, you should be able to tell where f' is positive or negative and increasing or decreasing, but the concavity of f' is not obvious. To see that the pieces of the graph of f' are linear, you would actually need to calculate f'.

- f' is undefined at $x = \pm 2$ because f is discontinuous at $x = \pm 2$. Here's another way to see why: we can go back to the definition of the derivative. For example, $f'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$; since $f(2)$ is undefined, the expression $\dfrac{f(2+h) - f(2)}{h}$ is also undefined, so $f'(2)$ is also undefined.

44. Gromit has been growing a giant squash for Tottington Hall's annual Giant Vegetable Competition. He has carefully tracked his squash's length and weight. Let $w(\ell)$ be the squash's weight in kg when its length is ℓ cm.

(a) *What are the units of $w'(\ell)$?*

Solution. $\boxed{\dfrac{\text{kg}}{\text{cm}}}$. The derivative is the rate of change of $w(l)$ with respect to l and we know the units of $w(l)$ are in kg and the units of l are cm.

(b) *Another notation for $w'(\ell)$ is $\dfrac{dw}{d\ell}$. Do you expect $\dfrac{dw}{d\ell}$ to be positive or negative? Why?*

Solution. $\dfrac{dw}{d\ell}$ should be positive. As the length of the squash increases, the mass should increase.

(c) Suppose $w'(70) = 8$. (This statement can also be written as $\left.\dfrac{dw}{d\ell}\right|_{\ell=70} = 8$.) Which is the following is the most reasonable conclusion?

A. It takes 70 days for the squash to grow to be 8 kg.

B. When the squash is 70 cm long, it weighs 8 kg.

C. When the squash is 70 cm longer than its current length, it will weigh 8 kg more than it currently does.

(D.) When the squash grows from 70 cm to 71 cm, it will gain about 8 kg in weight. This one.

(d) *Interpret the statement $w'(105) = 10$ in words.*

Solution. When the squash grows from 105 cm to 106 cm, it will gain *about* 10 kg in mass.

45. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \left(3e^u - u^2 + 5\right) du$

Solution. $\displaystyle\int \left(3e^u - u^2 + 5\right) du = \boxed{3e^u - \frac{u^3}{3} + 5u + C}$.

(b) $\displaystyle\int \frac{5}{7x}\, dx$

Solution. First, this is an indefinite integral, so it's really asking for all antiderivatives of $\dfrac{5}{7x}$. Let's rewrite this: $\displaystyle\int \frac{5}{7x}\, dx = \int \frac{5}{7} \cdot \frac{1}{x}\, dx = \boxed{\frac{5}{7} \ln|x| + C}$.

(c) $\displaystyle\int (e^\pi + ey + \pi^y)\, dy$

Solution. $\displaystyle\int (e^\pi + ey + \pi^y)\, dy = \boxed{e^\pi y + \frac{e}{2} y^2 + \frac{\pi^y}{\ln \pi} + C}$. (Keep in mind that e^π is a constant.)

(d) $\displaystyle\int (\pi + x)\sqrt{x}\, dx$

Solution. We don't have a rule about integrating products, so let's first rewrite the integrand so that it's no longer a product:

$$\begin{aligned}\int (\pi + x)\sqrt{x}\,dx &= \int (\pi\sqrt{x} + x\sqrt{x})\,dx \\ &= \int \left(\pi x^{1/2} + x^{3/2}\right) dx \\ &= \boxed{\frac{2}{3}\pi x^{3/2} + \frac{2}{5}x^{5/2} + C}\end{aligned}$$

(e) $\displaystyle\int \left(\frac{e}{x^2} + 2^x \cdot 3^x\right)\,dx$

Solution. We don't have a rule about integrating products, so we must first rewrite $2^x \cdot 3^x$:

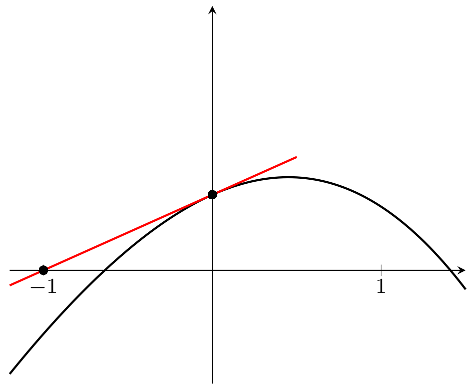
$$\begin{aligned}\int \left(\frac{e}{x^2} + 2^x \cdot 3^x\right) dx &= \int \left(ex^{-2} + 6^x\right)\,dx \\ &= -ex^{-1} + \frac{1}{\ln 6}6^x + C \\ &= \boxed{-\frac{e}{x} + \frac{1}{\ln 6}6^x + C}\end{aligned}$$

46. (a) *Use Newton's Method to estimate a solution to* $\sin(x) = x^2 - 1$*. Start with* $x_0 = 0$ *and calculate* x_1 *and* x_2*.* Note: *One of these you can compute by hand, for the other you'll need a calculator.*

Solution. We want $\sin(x) = x^2 - 1 \Rightarrow \sin x - x^2 + 1 = 0$. So let $f(x) = \sin x - x^2 + 1$. Then $f'(x) = \cos x - 2x$. By our formula,

$$\begin{aligned}x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 0 - \frac{f(0)}{f'(0)} \\ &= -\frac{\sin(0) - 0^2 + 1}{\cos(0) - 2(0)} \\ &= -\frac{1}{1} \\ &= -1\end{aligned}$$

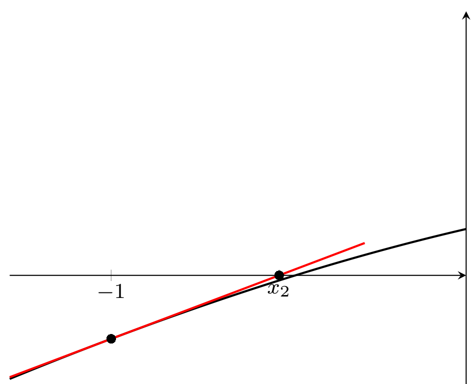
Here's the graph of $f(x) = \sin x - x^2 + 1$ with these approximations:



Now with $x_1 = -1$, we have that:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= -1 - \frac{f(-1)}{f'(-1)} \\ &= -1 - \frac{\sin(-1) - (-1)^2 + 1}{\cos(-1) - 2(-1)} \\ &\approx -0.6687516 \end{aligned}$$

Here's the graph of $f(x) = \sin x - x^2 + 1$ zoomed in with these approximations:



We see that we are really close to the actual root now. In fact $f(x_2) \approx -0.0672357$ which is very close to zero.

(b) *What would have happened if we'd started with $x_0 = 1$ instead of $x_0 = -1$?*

Solution. We can see from the picture that in this case there is no way we would have gotten close to the root we wanted. Instead we would have gotten close to the positive root.

47. Consider the function $f(x) = -3x^3 + 5x^2 - 1$.

(a) *Are you able to algebraically solve $f(x) = 0$?*

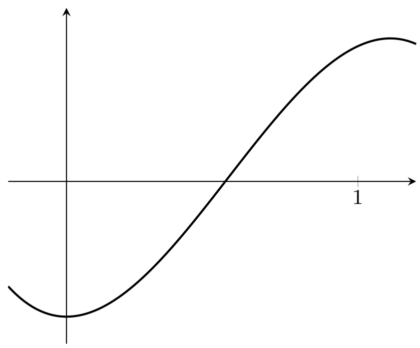
Solution. No! This is a cubic, and we have no formula that will allow us to solve this equation.

(b) *Does f have a root in the interval [0,1]? How can you tell? How many roots are there on the interval [0,1]?*

Solution. Since f is continuous on [0,1], we can use the Intermediate Value Theorem to show that a root exists. Note that $f(0) = -1 < 0$, and $f(1) = 1$ is clearly positive. So f must hit all outputs between -1 and 1. In particular, f must hit 0. And so there is at least one root on [0,1].

To check if there are multiple roots, we could use Rolle's Theorem like we did on #4 on the handout from Section 4.2. Another (easier) option is to check whether f is increasing or decreasing on [0,1]. Note that $f'(x) = -9x^2 + 10x$ which has no critical points in (0,1). In fact $f'(x) > 0$ for all x in (0,1). Thus f is increasing, and so it has exactly one root on [0,1].

(c) *Here is a graph of $f(x)$:*



From the graph, in order to estimate the root, it seems reasonable to start with $x_0 = 1$. Use the formula for Newton's method to find x_1 and x_2. Represent your work on the graph above!

Solution. Using Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note that $f(x) = -3x^3 + 5x^2 - 1$ and $f'(x) = 9x^2 + 10x$.

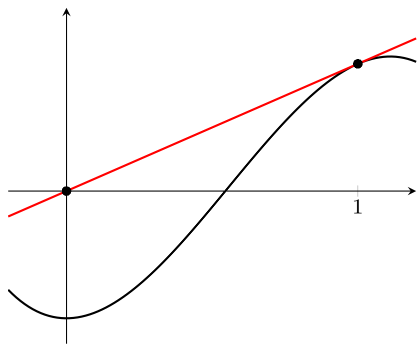
From the formula,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Since $x_0 = 1$:

$$\begin{aligned} x_1 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{-3+5-1}{-9+10} \\ &= 1 - 1 \\ &= \boxed{0} \end{aligned}$$

Here's a picture of what happened here:



Again, from the formula:

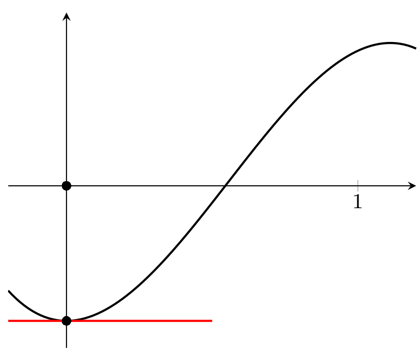
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Since $x_1 = 0$:

$$\begin{aligned} x_2 &= 0 - \frac{f(0)}{f'(0)} \\ &= 0 - \frac{0+0-1}{0+0} \end{aligned}$$

This does not exist because $f'(x_1) = f'(0) = 0$, and we can not divide by zero.

Note that algebraically this clearly does't make sense. Here is what happened graphically:



Since the tangent line at x_1 is horizontal, it will never meet the x-axis, and so we can not find the next approximation!

(d) *Does your work in (c) mean that you can not use Newton's Method here to approximate this root?*

Solution. Not necessarily! But it does mean that we need to change our initial approximation. It definitely fails if we start with $x_0 = 1$.

48. True/False

(a) Let $f(x)$ be continuous on the interval $[-1,3]$. If $f(-1) = 2$ and $f(3) = 8$, then by the Intermediate Value Theorem, $f(x)$ cannot have a zero on $[-1,3]$.

Solution. False.

(b) $\arccos(\cos(-\frac{\pi}{8})) = \frac{\pi}{8}$.

Solution. True. Since $\cos(x)$ is an even function, $\cos(-\frac{\pi}{8}) = \cos(\frac{\pi}{8})$. Since the range of $\arccos(x)$ is $[0,\pi]$, we have $\arccos(\cos(-\frac{\pi}{8})) = \frac{\pi}{8}$.

(c) Let f and g be two functions defined on $(-\infty, \infty)$, $f(0) \neq g(0)$, then we must have $\lim_{x\to 0} f(x) \neq \lim_{x\to 0} g(x)$.

Solution. False.

(d) If f is even and $\lim_{x\to 0^+} f(x)$ exists, then $\lim_{x\to 0} f(x)$ exists.

Solution. True.

(e) If $g(x)$ is an odd function and has a local maximum at $x = c$, then it must have a local minimum at $x = -c$.

Solution. True. Since $g(-x) = -g(x)$, g is symmetric about the origin. If $x = c$ is a local maximum, then for x close to c, $g(x) \leq g(c)$. So for x close to $-c$, $g(x) \geq g(-c)$. So $x = -c$ is a local minimum

(f) If $f(x)$ is concave down and $f'(a) = 0$, then f has a maximum $x = a$.

Solution. True, because of the second derivative test.

(g) If $f'(a) = 0$, then $f(x)$ has either a local maximum or a local minimum at $x = a$.

Solution. False, consider $f(x) = x^3$ and $a = 0$.

(h) If c is a local minimum of f, then $f''(c) > 0$.

Solution. False. f'' could be 0 or undefined at $x = 0$, making the second derivative test inconclusive.

(i) A function may have three different horizontal asymptotes.

Solution. False. We find horizontal asymptotes of a function $f(x)$ by computing $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$. Thus, a function can have at most two horizontal asymptotes.

(j) The function $|x - 2|$ is differentiable at $x = 2$.

Solution. False, the graph has a corner (is not locally linear) at $x = 2$.

(k) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x)$.

Solution. False. Consider $f(x) = x$ and $g(x) = x + 1$.

(l) If f is differentiable at $x = a$, then f is continuous at $x = a$.

Solution. True. Differentiability requires continuity in the definition.

(m) If f is continuous at $x = a$, then f is differentiable at $x = a$.

Solution. False. Corners (points where a function is not locally linear) are points where a function is continuous but not differentiable. For example $|x|$ is continuous but not differentiable at $x = 0$.

(n) There is a function f so that $f(x) > 0$ for all x but $f'(x) < 0$ for all x.

Solution. True. For example $f(x) = e^{-x}$. Or sketch any curve that decreases to a nonnegative horizontal asymptote.