Math 2551 Final Review Fall 2023

Math 2550 students should work problems 1-18 only.

1. Find $\mathbf{T}, \mathbf{N}$, and the curvature κ for $\mathbf{r}(t) = 3\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j} + 4t\mathbf{k}$.

2. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

3. **True** or **False**: Suppose $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = 5$ along the lines $x = a$ and $y = b$ and $f(a,b) = 5$. Then f must be continuous at (a,b).

4. Find all second-order partial derivatives for $w = x\sin(x^2 y)$

5. Evaluate $\dfrac{dw}{dt}$ for $w = 2ye^x - \ln(z)$ if $x(t) = \ln(t^2+1), y = \arctan(t), z = e^t$ at $t = 1$.

6. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the surface $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1,1,1)$.

7. Based on the contour plot below for a function $f(x,y)$, determine the sign $(+,-,0)$ of

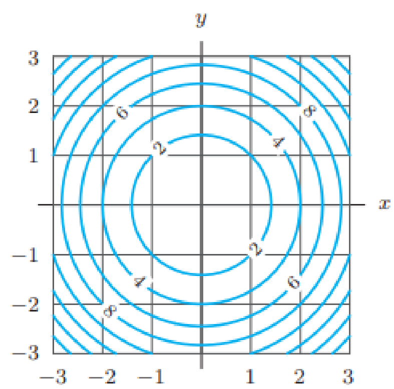
   (a) $f_x(0,2)$

   (b) $f_y(2,-2)$

   (c) the rate of change of f at $(-2,0)$ in the direction towards $(-3,3)$

   (d) $Df_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle}(-1,-1)$

   (e) the rate of change of f in the direction tangent to the level curve through $(0,1)$



8. Find the derivative of the function $f(x,y) = 2xy - 3y^2$ in the direction $4\mathbf{i} + 3\mathbf{j}$ at the point $(5,5)$.

9. Find the tangent plane and normal line at $(2,0,2)$ of the surface $2z - x^2 = 0$.

10. Find all local maxima, local minima, and saddle points for $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$.

11. Find the absolute maximum and minimum of $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular region bounded by the lines $x = 0, y = 2$, and $y = 2x$.

12. Find the maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

13. Write an iterated integral using (a) vertical cross-sections and (b) horizontal cross-sections whose region of integration is the region bounded by $y = \sqrt{x}, y = 0, x = 9$.

14. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$.

15. Change to polar coordinates and evaluate the integral $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy\, dx$.

16. Integrate the function $f(x,y,z) = 3 - 4x$ over the region below $z = 4 - xy$ and above the rectangle $0 \le x \le 2, 0 \le y \le 1$ in the xy-plane.

17. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

18. Use the change of coordinates $x(u, v) = \dfrac{u}{v}, y(u, v) = uv$ to evaluate the integral $\displaystyle\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dA$, where R is the region in the first quadrant bounded by $xy = 1, xy = 9, y = x$, and $y = 4x$.

19. Evaluate the line integral $\int_C (xy + y + z)\ ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}$ for $0 \leq t \leq 1$.

20. Find the flow of the field $\mathbf{F} = \langle -4xy, 8y, 2\rangle$ along the curve $\mathbf{r}(t) = \langle t, t^2, 1\rangle, 0 \leq t \leq 2$.

21. Find the counterclockwise circulation and the outward flux of the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ around/through the unit circle centered at the origin.

22. Find the potential function for $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$.

23. Use Green's Theorem to find counterclockwise circulation and outward flux of the field $\mathbf{F} = \langle y^2 - x^2, x^2 + y^2\rangle$ for the curve C enclosing the region bounded by $y = 0, x = 3$, and $y = x$.

24. Let f be a function of three variables and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields in $\mathbb{R}^3$. Which of the following expressions make mathematical sense? If you can compute any of them, do so.

    (a) $\text{curl}(\text{div}(\mathbf{F}))$

    (b) $\text{curl}(\nabla f)$

    (c) $\text{div}(\mathbf{F} \cdot \mathbf{G})$

    (d) $\text{curl}(\text{div}(f))$

    (e) $\text{div}(\text{curl}(\mathbf{G}))$

    (f) $\text{div}(\nabla f)$

25. Use a parameterization to write a double integral for the area of the surface S which is the portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$.

26. Evaluate $\iint_S 2y\ d\sigma$ over the surface S which is the part of the cylinder $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

27. Let S be the surface that consists of the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane and below the cone $z = 3\sqrt{x^2 + y^2}$.

    (a) Sketch S.

    (b) Find a parameterization of S.

    (c) Calculate the area of S.

28. Let S be the surface consisting of the top half $z \geq 0$ of the sphere $x^2 + y^2 + z^2 = 9$, together with the disk $x^2 + y^2 \leq 9, z = 0$, its base in the xy-plane. Use the Divergence Theorem to evaluate $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma$, where $\mathbf{F}(x, y, z) = 3xy^2\mathbf{i} + 3x^2y\mathbf{j} + z^3\mathbf{k}$.

29. Let S be the part of the surface $z = 4x^2 + y^2 - 4$ beneath the plane $z = 5$. Let C be the bounding curve of S in the plane $z = 5$, traversed counterclockwise and suppose S is oriented accordingly (normals towards the z-axis). Let $\mathbf{F}(x, y, z) = \langle 2y, 4x, e^x\rangle$. Use Stokes' Theorem to evaluate the curl integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ d\sigma.$$

30. Let S be the surface of the cylinder defined by $y^2 + z^2 = 4$ between the planes $x = -1$ and $x = 3$. Let $\mathbf{F}(x, y, z) = e^{xy}\mathbf{i} + e^x y\mathbf{j} + e^x z\mathbf{k}$.

    (a) Sketch S.

    (b) Find a parameterization of S.

(c) Let $\mathbf{n}$ be an outward pointing unit normal for S. Evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

by direct calculation (do not use the Divergence Theorem).

Math 2551 Final Review Fall 2023 ANSWERS

1. $\mathbf{T} = \langle 3\cos(t)/5, -3\sin(t)/5, 4/5\rangle, \mathbf{N} = -\sin(t)\mathbf{i} - \cos(t)\mathbf{j}, \kappa = \frac{3}{25}$
2. $5/2$
3. **False**
4. $w_{xx} = 6xy\cos(x^2y) - 4x^3y^2\sin(x^2y),\ w_{yy} = -x^5\sin(x^2y),\ w_{xy} = w_{yx} = 3x^2\cos(x^2y) - 2x^4y\sin(x^2y)$
5. $\frac{dw}{dt} = 4t\arctan(t) + 1 \mid_{t=1} = \pi + 1$
6. $\dfrac{\partial z}{\partial x} = \dfrac{1}{4},\ \dfrac{\partial z}{\partial y} = \dfrac{-3}{4}$
7. (a) 0
   (b) $-$
   (c) $+$
   (d) $-$
   (e) 0
8. -4
9. $2x - z - 2 = 0, \mathbf{r}(t) = \langle 2, 0, 2\rangle + t\langle -4, 0, 2\rangle$
10. $f(-3, 3) = -5$ is a minimum
11. $f(0, 0) = 1$ is the abs. max, $f(1, 2) = -5$ is the abs. min
12. 39
13. Regions of integration are a) $0 \le x \le 9, 0 \le y \le \sqrt{x}$, b) $0 \le y \le 3, y^2 \le x \le 9$.
14. $e - 2$
15. $\frac{\pi}{2}$
16. $\int_0^2 \int_0^1 \int_0^{4-xy} (3 - 4x)\ dz\ dy\ dx = -\frac{17}{3}$
17. $\frac{4\pi}{3}$
18. $\int_1^2 \int_1^3 \frac{(u+v)2u}{v}\ du\ dv = 8 + \frac{52}{3}\ln(2)$
19. $13/2$
20. 48
21. Circulation $= 0$, Flux $= 2\pi$
22. $f(x, y, z) = xe^{y+2z} + C$
23. Circulation $= 9$, Flux $= -9$
24. (a) Does not make sense
    (b) 0
    (c) Does not make sense
    (d) Does not make sense
    (e) 0
    (f) $f_{xx} + f_{yy} + f_{zz}$
25. $\int_0^{2\pi} \int_1^3 r\sqrt{5}\ dr\ d\theta$

26. $\mathbf{r}(x,\theta) = \langle x, 2\sin(\theta), 2\cos(\theta)\rangle$,

$$\iint_R f(\mathbf{r}(x,\theta))|\mathbf{r}_x \times \mathbf{r}_\theta|\ dA = \int_0^{2\pi}\int_0^{3-2\cos(\theta)} 2(2\sin(\theta))(2)\ dx\ d\theta = 0$$

27. (b) $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 4-r^2\rangle, 1 \le r \le 2, 0 \le \theta \le 2\pi$ (c) $\frac{\pi}{6}(17^{3/2} - 5^{3/2})$

28. $\frac{1458}{5}\pi$

29. 9π

30. (b) $\mathbf{r}(x,\theta) = \langle x, 2\cos(\theta), 2\sin(\theta)\rangle, -1 \le x \le 3, 0 \le \theta \le 2\pi$, (c) $8\pi(e^3 - e^{-1})$