MATH 1554 READING DAY STUDY SESSION WORKSHEET

Problems

1. A 5×4 matrix $A = [\vec{a_1} \quad \vec{a_2} \quad \vec{a_3} \quad \vec{a_4}]$ has all non-zero columns, and $\vec{a_4} = 2\vec{a_1} + 3\vec{a_2} + 5\vec{a_3}$. Find a non-trivial solution to $A\vec{x} = \vec{0}$.

2. For what values of h, if any, are the columns of A linearly dependent? $A = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & 1 \\ h & 1 & 0 \end{bmatrix}$

3. For what values of h is $\vec{b}$ in the plane spanned by $\vec{a_1}$ and $\vec{a_2}$?
$$\vec{a_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{a_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ h \end{bmatrix}$$

4. Express the solution to $A\vec{x} = \vec{0}$ in parametric vector form, where $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

5. Write down the standard matrix A of $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(\vec{x}) = -\vec{x}$.

6. Find the domain and co-domain of the linear transformation T given by the standard matrix
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 7 & 3 \\ 2 & 5 & -1 \end{bmatrix}$$
Is this linear transformation one-to-one? Is it onto?

7. Let $A = \begin{bmatrix} -5 & 2 \\ -1 & -3 \end{bmatrix}$. Find its eigenvalue(s) and find an invertible matrix P and a (rotation-scaling) matrix C such that $A = PCP^{-1}$.

8. W is the set of all vectors of the form $\begin{bmatrix} x \\ x+y \\ y \end{bmatrix}$. Which of the following vectors are in $W^{\perp}$?
$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

9. Identify all values of a, b, c, if any, so that the columns of U are mutually orthogonal, $U = \begin{bmatrix} 3 & 2 & 2 \\ -4 & 1 & b \\ 2 & a & c \end{bmatrix}$.

10. Use the Gram-Schmidt process to construct an orthonormal basis of the column space of $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ -1 & -4 \end{bmatrix}$.

11. Let $A = QR$, where $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ -2 & -2 \end{bmatrix}$, $Q = \dfrac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}$ Compute the upper triangular matrix R.

12. Give an example of a 2×2 matrix that is in echelon form, is orthogonally diagonalizable, but is not invertible.

13. Make a change of variable, $\vec{x} = P\vec{y}$, that transforms $Q(\vec{x}) = 3x_1^2 + 4x_1x_2$ into a form that does not have cross-product terms. Give P and the new quadratic form.

14. Construct a SVD of matrix $A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

15. True or False?

(i) If the set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, so is every pair of vectors $\{\vec{u}, \vec{v}\}$, $\{\vec{u}, \vec{w}\}$, and $\{\vec{v}, \vec{w}\}$.

(ii) If every pair of vectors $\{\vec{u}, \vec{v}\}$, $\{\vec{u}, \vec{w}\}$, and $\{\vec{v}, \vec{w}\}$ is linearly independent, so is the set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$.

(iii) For any two vectors $\vec{u}$ and $\vec{v}$, we have $Span\{\vec{u}, \vec{v}\} = Span\{\vec{u}, 2\vec{u} + 3\vec{v}, 4\vec{v}\}$.

(iv) If $\vec{u}$ and $\vec{v}$ are two distinct nonzero vectors, then there are exactly to vectors in $Span\{\vec{u}, \vec{v}\}$.

(v) The transformation given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1x_2 \\ x_2 \end{bmatrix}$ is linear.

(vi) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a projection onto the x_1-axis. The range of T is $\mathbb{R}^2$.

(vii) The transformation given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix}$ is linear.

(viii) A linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ can be onto.

(ix) The composition $S \circ T$ of two one-to-one linear maps is one-to-one.

(x) The range of a one-to-one linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ may be a line.

(xi) The eigenvalues of a square matrix A are the same as the eigenvalues of its reduced row echelon form.

(xii) If $\vec{u}$ and $\vec{v}$ are eigenvectors corresponding to the same eigenvalue λ, then every linear combination of $a\vec{u} + b\vec{v}$ with $a, b \in \mathbb{R}$ (except the zero vector) is an eigenvector.

(xiii) The geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity.

(xiv) All upper triangular 3×3 stochastic matrices are not regular.

(xv) If A is a diagonalizable matrix, then $\lambda = 0$ is not an eigenvalue of A.

(xvi) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

(xvii) If complex λ is an eigenvalue, then so is $-\lambda$.

(xviii) Every real 3×3 matrix must have a real eigenvalue.

(xix) For any three vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ we have $(\vec{x} \cdot \vec{y})\vec{z} = (\vec{y} \cdot \vec{z})\vec{x}$.

(xx) Let $\vec{x} \cdot \vec{y} > 0$. Then the angle between $\vec{x}$ and $\vec{y}$ is less than $90°$.

(xxi) Every orthogonal set of nonzero vectors $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.

(xxii) Let $\hat{y}$ be the orthogonal projection of a vector $\vec{y}$ onto the subspace $W \subset \mathbb{R}^n$. Then the transformation $T(\vec{y}) = \hat{y}$ is linear.

(xxiii) The inverse of an orthogonal matrix Q is Q^T.

(xxiv) A least square solution $\hat{x}$ to $A\vec{x} = \vec{b}$ always satisfies $A\vec{x} = \vec{b}$.

(xxv) A least square solution $\hat{x}$ to $A\vec{x} = \vec{b}$ minimizes the distance $||A\vec{x} - \vec{b}||$. That is, the distance is the shortest for $\vec{x} = \hat{x}$.

(xxvi) A $n \times n$ symmetric matrix A will always have n real and distinct eigenvalues.

(xxvii) A $n \times n$ symmetric matrix A will have algebraic multiplicity = geometric multiplicity for each of its eigenvalues.

(xxviii) If a matrix A is orthogonally diagonalizable, then A^k is also orthogonally diagonalizable for all $k \in \mathbb{Z}^+$.

(xxix) The eigenvalues of A^TA are always real for any $m \times n$ matrix A.

(xxx) A negative definite matrix cannot be invertible.

(xxxi) Matrices A and A^T have the same non-zero singular values.

(xxxii) For any matrix A, A^tA has non-negative, real eigenvalues.

(xxxiii) The maximum value of the quadratic form, $Q = 8x_1^2 + 4x_2^2 + x_3^2$, for any $\vec{x} \in \mathbb{R}^3$, is 8.

(xxxiv) If the number of non-zero singular values of a square matrix A equals the number of its columns, then A is invertible.

(xxxv) If A is an orthogonal matrix, then the largest singular value of A is 1.

Answers

1. $\vec{x} = (2, 3, 5, -1)$

2. None. The columns of A are linearly independent.

3. $h = 3$

4. $\vec{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.$

5. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

6. Domain: $\mathbb{R}^3$, Codomain: $\mathbb{R}^4$, Not one-to-one, Not onto.

7. $\lambda = -4 \pm i$, $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -4 & -1 \\ 1 & -4 \end{bmatrix}$

8. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

9. $a = -1$, $b = 7$, $c = 11$

10. $\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}$, $col(A) = Span\left\{ \frac{\vec{v_1}}{\sqrt{6}}, \frac{\vec{v_2}}{2\sqrt{2}} \right\}$

11. $R = \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

13. Eigenvalues are $-1, 4$, $P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, $Q = 4y_1^2 - y_2^2$.

14. $U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)^T$

15. True/false:

(i) True
(ii) False
(iii) True
(iv) False
(v) False
(vi) False
(vii) False
(viii) False
(ix) True
(x) False
(xi) False
(xii) True
(xiii) True
(xiv) True
(xv) False
(xvi) True
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(xxvii) True
(xxviii) True
(xxix) True
(xxx) False
(xxxi) True
(xxxii) True
(xxxiii) False
(xxxiv) False
(xxxv) True: $\sigma_1 = ||A\vec{v}_1|| = ||\vec{v}_1|| = 1$