Reading Day Problem Bank
Chapters 1-4

Differentiation Rules to Remember!

- **Derivative of a Constant:** $\dfrac{d}{dx}(c) =$

- **Power Rule:** $\dfrac{d}{dx}(x^n) =$

- **Constant Multiple Rule:** If c is a constant, then $\dfrac{d}{dx}[cf(x)] =$

- **Sum Rule:** $\dfrac{d}{dx}[f(x) + g(x)] =$

- **Product Rule:** $\dfrac{d}{dx}[f(x) \cdot g(x)] =$

- **Quotient Rule:** $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] =$

- **Derivative of the Natural Exponential Function:** $\dfrac{d}{dx}(e^x) =$

Derivatives of the six basic trigonometric functions.

$\dfrac{d}{dx}(\sin x) =$ $\qquad\qquad$ $\dfrac{d}{dx}(\cos x) =$ $\qquad\qquad$ $\dfrac{d}{dx}(\tan x) =$

$\dfrac{d}{dx}(\csc x) =$ $\qquad\qquad$ $\dfrac{d}{dx}(\sec x) =$ $\qquad\qquad$ $\dfrac{d}{dx}(\cot x) =$

The Chain Rule

If $f(u)$ is differentiable at u, and $u = g(x)$ is differentiable at x, then the derivative of the composite function $f \circ g(x) = f(g(x))$, with respect to x, is

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{OR} \quad \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Derivatives of Log and Exponential Functions If $b > 0$, then

- $\frac{d}{dx}\left(\log_b x\right) =$

- $\frac{d}{dx}\left(b^x\right) =.$

Derivatives of the inverse trigonometric functions.

$\dfrac{d}{dx}(\arcsin x) =$ $\qquad\qquad$ $\dfrac{d}{dx}(\arccos x) =$ $\qquad\qquad$ $\dfrac{d}{dx}(\arctan x) =$

1. Evaluate the following derivatives; you need not simplify your answers.

   (a) $\frac{d}{dx}\left[x^2 \log_3(2x+1)\right]$

   (b) If $y = (\tan 2)e^{\sqrt{\cos t}}$, find $\frac{dy}{dt}$.

   (c) If $g(x) = \sec(x^2 e^x)$, find $g'(x)$.

   (d) If $f(t) = \dfrac{t}{5\sqrt[3]{1+e^t}}$, find $f'(t)$. (Can you rewrite f so that you don't need to use the Quotient Rule?)

   (e) $f(x) = x\sin\left(x^3\right)$, $g(x) = x\sin^3 x$, and $h(x) = x(\sin x)^3$. (Are these the same or different?)

2. Compute derivatives for the following functions:

   (a) $f(x) = \ln(\sec(\ln x))$.

   (b) $f(x) = \arccos(e^{3-x^2})$.

   (c) $f(x) = (\ln x)^{\ln x}$.

   (d) $f(x) = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$.

   (e) $u(x) = \sin\left(\tan^{-1}(\ln x)\right)$

   (f) $a(t) = \dfrac{(t+2)^3(t+5)^7}{\sqrt{t-5}}$

   (g) $s(t) = \log_2\left(\log_3\left(e^{t^2}\right)\right)$

   (h) $h(x) = \dfrac{\sqrt{\arctan(4x)}}{e^5}$

   (i) $f(t) = 3^t + t^3 + (\ln t)^{3t}$

   (j) $f(x) = \log_3\left(\arctan(x)\arcsin(x)\right)$.

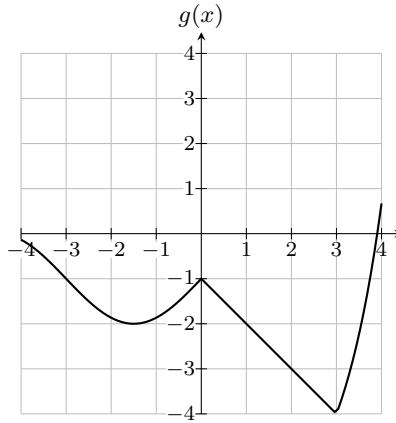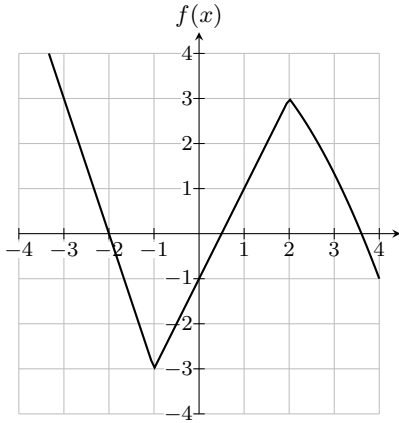3. Find the equation of the tangent line to the curve

   $$e^{2x} = \sin(x^2 + 2y) + 1.$$

   at the point (0,0).

4. Find the line tangent to the heart curve $(x^2 + y^2 - 1)^3 - x^2 y^3 = 0$ at $(-1, 1)$.
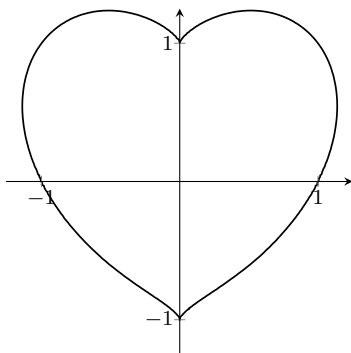
5. Here are graphs of two functions, $f(x)$ and $g(x)$. If $F(x) = f(g(x))$, what is $F'(1)$?

6. You are given the following information about three functions f, g, and h.

$$h(1) = 2 \qquad\qquad g(2) = 3 \qquad\qquad f'(3) = 6$$
$$h'(1) = 4 \qquad\qquad g'(2) = 5$$

If $r(x) = f(g(h(x)))$, do you have enough information to find $r'(1)$? If so, compute it. If not, what additional information do you need?

7. Look again at the folium of Descartes, $x^3 + y^3 = \dfrac{9}{2}xy$. Find all points on the curve where the tangent line is horizontal.
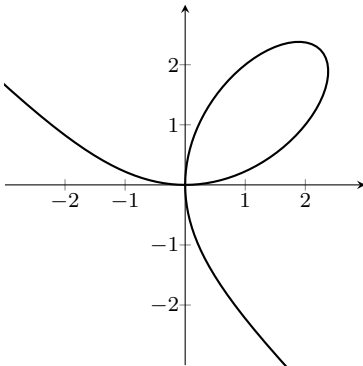
8. Suppose you want to compute derivatives of the following functions. For which would you want to use logarithmic differentiation? Which could you do without logarithmic differentiation? You do not need to actually compute these derivatives!

(a) $(x^3 + 17\sin x)^{\ln 3}$
  (b) $7^{\sqrt{x^3 + 14}}$
  (c) $(x^2 + \sqrt{x})^{\tan x}$

9. Two cars are approaching an intersection. A red car, approaching from the north, is traveling 20 feet per second and is currently 60 feet from the intersection. A blue car, approaching from the west, is traveling 30 feet per second and is currently 80 feet from the intersection. At this moment, is the distance between the two cars increasing or decreasing? How quickly?

10. An oil tank in the shape of an inverted cone has height 10 m and radius 6 m. When the oil is 5 m deep, the tank is leaking oil from the tip at a rate of 2 m^3 per day. How quickly is the height of the oil in the tank decreasing at this moment?

Note: The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

11. At noon, you are running to get to class and notice a friend 100 feet west of you, also running to class. If you are running south at a constant rate of 450 ft/min (approximately 5 mph) and your friend is running north at a constant rate of 350 ft/min (approximately 4 mph), how fast is the distance between you and your friend changing at 12:02 pm?

12. During a night run, an observer is standing 80 feet away from a long, straight fence when she notices a runner running along it, getting closer to her. She points her flashlight at him and keeps it on him as he runs.

    When the distance between her and the runner is 100 feet, he is running at 9 feet per second. At this moment, at what rate is she turning the flashlight to keep him illuminated? Include units in your answer.

13. As you're riding up an elevator, you spot a duck on the ground, waddling straight towards the base of the elevator. The elevator is rising at a speed of 10 feet per second, and the duck is moving at 5 feet per second towards the base of the elevator. As you pass the eighth floor, 100 feet up from the level of the river, the duck is 200 feet away from the base of the elevator. At this instant, at what rate is the distance between you and the duck changing?

14. Kelly is flying a kite; the kite is 100 ft above the ground and moving horizontally away from Kelly. At precisely 1 pm, Kelly has let out 300 ft of string, and the amount of string let out is increasing at a rate of 5 ft/s. If Kelly is standing still, at what rate is the angle between the string and the vertical increasing at 1 pm? (You may assume that the string is stretched taut so that it is a straight line.)

15. A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at a rate of 5 ft/sec.

    (a) How fast is the top of the ladder sliding down the wall then?

    (b) At what rate is the area of the triangle formed by the ladder, wall and ground changing then?

    (c) At what rate is the angle between the ladder and the ground changing then?

16. (a) Use linear approximation to estimate $\sqrt{24.5}$ without using a calculator. Draw a sketch to explain what you are doing.

    (b) From your sketch, you should be able to tell whether your approximation is an overestimate or underestimate. Which is it?

    (c) Explain your answer to (b) using a second derivative.

17. Use linear approximation to estimate $9^{4/3}$. Is your estimate too high or too low?

18. In this problem, we'll look at the cubic function $f(x) = x^3 + 3x^2 + 1$.

    (a) Find all critical points of f.

    (b) Make a sign chart for f', and use this to decide whether each of the critical points you found is a local minimum, a local maximum, or neither.

19. Let $f(x) = x^4 - 8x^2 + 16$.

(a) Find all critical points of f.

(b) For each critical point c of f, find the sign of $f''(c)$. What does this tell you about the critical point c?

20. Suppose you've made the following sign chart for the derivative of a function $f(x)$. The function $f(x)$ is continuous and differentiable on $(-\infty, \infty)$.

$$\begin{array}{c}
\hline
\quad \overset{+}{0} \quad \overset{+}{3} \quad \overset{+}{5} \\
\text{sign of } f' \quad + \qquad - \qquad + \qquad -
\end{array}$$

(a) Does $f(x)$ have an absolute maximum on $(-\infty, \infty)$? (Definitely yes, definitely no, or maybe?) If so, where could it be?

(b) Does $f(x)$ have an absolute minimum on $(-\infty, \infty)$? If so, where could it be?

(c) Does $f(x)$ have an absolute maximum on $[-2, 10]$? If so, where could it be?

(d) Does $f(x)$ have an absolute minimum on $[-2, 10]$? If so, where could it be?

21. Let $f(x) = 3x^{1/3} + 4x$. Find all critical points of f, and determine whether each critical point is a local minimum, local maximum, or neither.

22. *Sanity Check.* Do this without looking at your notes! Which of the following is the linearization of a function $f(x)$ at $x = a$?

A. $y = f(a) + f'(x)(x - a)$ E. $y = f'(a) + f(x)(x - a)$

B. $y = f(x) + f'(a)(x - a)$ F. $y = f'(x) + f(a)(x - a)$

C. $y = f(x) + f'(x)(x - a)$ G. $y = f'(x) + f(x)(x - a)$

D. $y = f(a) + f'(a)(x - a)$ H. $y = f'(a) + f(a)(x - a)$

23. (a) Use linear approximation to approximate the value of $\cos\left(\frac{93}{180}\pi\right)$ and $\cos\left(\frac{86}{180}\pi\right)$.

(b) To determine if the estimates are over or under the actual values, it is helpful to determine the concavity of the function. On the interval $[0, \pi]$ where is $f(x) = \cos(x)$ concave up? On the interval $[0, \pi]$ where is $\cos(x)$ concave down? Justify your answer using calculus.

(c) Are your estimates overestimates or underestimates? Sketch a graph to justify.

24. Use linear approximation to estimate $e^{0.1}$. Explain whether your estimate is overestimate of under estimate.

25. Use linear approximation to estimate $\sin^2(\frac{\pi}{4} - 0.1)$.

26. We'd like to get an accurate graph of $f(x) = \dfrac{\pi x^2 + 2}{x}$. Answer the questions below in whatever order makes most sense to you, and use them to sketch the graph of f.

(a) What is the domain of f? Does f have any useful symmetry?

(b) Find and classify all discontinuities of f. Justify your classifications using limits.

(c) Find all horizontal asymptotes of f.

(d) On what intervals is $f(x)$ increasing? On what intervals is it decreasing?

(e) On what intervals is $f(x)$ concave up? On what intervals is it concave down?

(f) What are the inflection points of f?

(g) Can you find the x- and y-intercepts of f?

(h) Sketch a rough graph of $f(x)$ that incorporates all of the above information.

27. Let $f(x) = 3x^{5/3}(x - 8)$.

Answer the questions below in whatever order makes most sense to you, and use them to sketch the graph of f.

(a) What is the domain of f? Does f have any useful symmetry?

(b) Find and classify all discontinuities of f. Justify your classifications using limits.

(c) Find all horizontal asymptotes of f.

(d) On what intervals is $f(x)$ increasing? On what intervals is it decreasing?

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(f) What are the inflection points of f?

(g) Can you find the x- and y-intercepts of f?

(h) Sketch a rough graph of $f(x)$ that incorporates all of the above information.

28. Graph $f(x) = \dfrac{x}{\sqrt{4 - x^2}}$. Show how you found the domain, the intervals on which the function is increasing and decreasing, the intervals on which the function is concave up and concave down, the local maxima and minima, any asymptotes, and anything else of interest on the curve.

29. Let $f(x) = \arcsin(4x^2)$.

(a) What is the domain of $f(x)$?

(b) Is $f(x)$ an even function, an odd function, or neither?

(c) On what intervals is f increasing? On what intervals is it decreasing?

(d) Does f have an absolute maximum and absolute minimum on its domain? If so, find the absolute maximum and minimum values, and say where they occur.

(e) In (a), you should have found that the domain of f was a closed interval $[a, b]$. What are $\lim\limits_{x \to a^+} f'(x)$ and $\lim\limits_{x \to b^-} f'(x)$? (Note that this is asking about f', not about f.) What does this tell you about the graph of f?

(f) Use the above information to sketch a rough graph of f. (Your sketch need not accurately reflect the concavity of the graph.)

30. Determine how many roots $f(x) = x^3 + x - 2$ has, if any, on the interval x in $[0,3]$.

31. Show that $f(x) = x + \ln(x)$ has a zero.

32. *Average speed vs. average velocity.* A swimmer is swimming a 100 m long race, which is one lap in a 50 m long pool. Let $s(t)$ be his distance from the starting position t seconds after the start of the race.

    (a) Which of the following is a more reasonable graph for $s(t)$? Why?


(A)

(B)
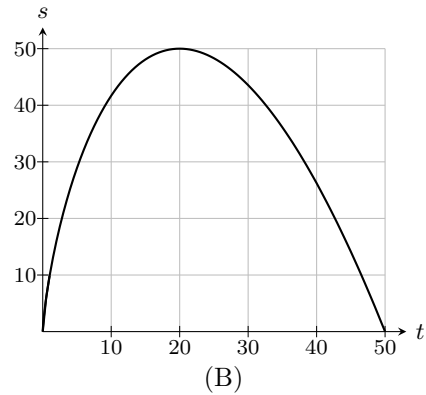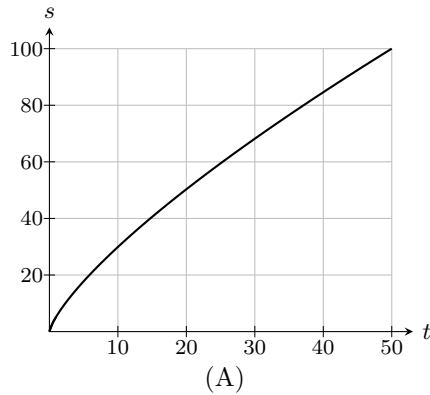
    (b) According to the graph you chose, what was the swimmer's average speed for the race? Average velocity for the race?

    (c) What was the swimmer's average speed over the first 20 seconds of the race? Average velocity?

    (d) What was the swimmer's average speed over the last 50 m of the race? Average velocity?

    (e) What is the difference between velocity and speed?

33. The graphs of f and g are given below.
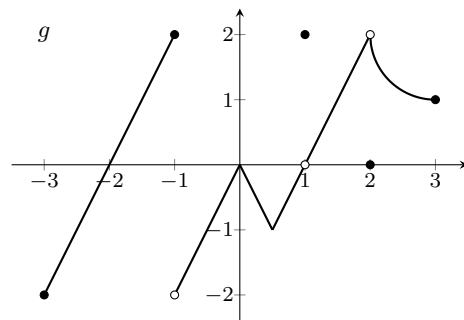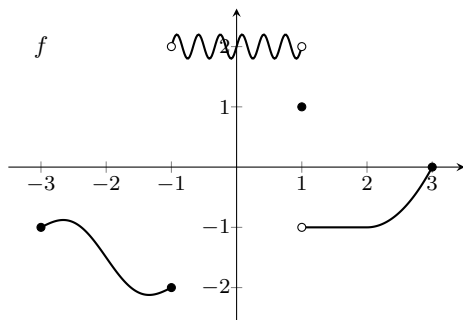
Evaluate each of the following limits.

(a) $\lim\limits_{x\to 2}(g(x)-f(x))$

(d) $\lim\limits_{x\to 1}\dfrac{f(x)}{g(x)}$

(b) $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$

(e) $\lim\limits_{x\to 2}\dfrac{f(x)}{g(x)}$

(c) $\lim\limits_{x\to -2}\dfrac{f(x)}{g(x)}$

34. *Working algebraically with limits.* Can you evaluate the following limits?

(a) $\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\sin x}{\cos x}$

(c) $\lim\limits_{x\to 3\pi^+}\dfrac{\cos x}{\sin x}$

(e) $\lim\limits_{x\to 2}\dfrac{(\sin x)-2}{(x-2)^2}$

(b) $\lim\limits_{x\to 3\pi^+}\dfrac{\sin x}{\cos x}$

(d) $\lim\limits_{x\to 3\pi}\dfrac{\cos x}{\sin x}$

(f) $\lim\limits_{x\to 3^+}\ln(x^2-9)$

35. In this problem, we'll look at $\lim\limits_{x\to 0}x^2\sin\left(\dfrac{\pi}{x}\right)$.

   (a) Aberforth is thinking about this limit and says, "As $x\to 0$, x^2 approaches 0. 0 times anything is 0, so $\lim\limits_{x\to 0}x^2\sin\left(\dfrac{\pi}{x}\right)$ must be 0." What do you think of Aberforth's reasoning?

   (b) Find $\lim\limits_{x\to 0}x^2\sin\left(\dfrac{\pi}{x}\right)$. Explain your reasoning carefully.

   (c) Sketch a rough graph of $f(x)=x^2\sin\left(\dfrac{\pi}{x}\right)$ for x near 0.

36. Evaluate the limits below. If the limit does not exists, write whether it is ∞ or $-\infty$. If it does not exists and also not $\pm\infty$, explain why the limit does not exist

(a) $\lim\limits_{x\to 0}\dfrac{3x-x\cos x}{\sin 2x}$.

(e) $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-3x+2}$

(h) $\lim\limits_{x\to -\infty}\dfrac{e^{2x}+e^x+4}{7e^{2x}+e^x-4}$

(b) $\lim\limits_{x\to 2}\dfrac{|x-2|}{x-2}$.

(f) $\lim\limits_{x\to\infty}\dfrac{\sqrt{2x^2+2x+3}}{3x-1}$

(i) $\lim\limits_{x\to\infty}\dfrac{70^{x/2}-1,000,000}{2^{3x}+3^{2x}+10^{100}}$

(c) $\lim\limits_{x\to\infty}2x-\sqrt{4x^2+1}$.

(d) $\lim\limits_{x\to 0}\csc(2x)\tan(3x)$

(g) $\lim\limits_{x\to\infty}\dfrac{e^{2x}+e^x+4}{7e^{2x}+e^x-4}$

(j) $\lim\limits_{x\to -\infty}\sin\left(\dfrac{1}{x}\right)$

37. Let a be a constant and f be the function defined by

$$f(x)=\begin{cases}\frac{ax}{6} & \text{for } x<1\\ \frac{1}{x+a} & \text{for } x\ge 1\end{cases}.$$

   Find all values of a for which f is continuous on $(-\infty,\infty)$. (Do this without using a calculator!)

38. Let f be the function defined by

$$f(x)=\begin{cases}-x^2+1 & \text{for } x\le 1\\ ax+b & \text{for } x>1\end{cases}$$

(a) For what values of a and b will f be continuous at $x = 1$?

(b) Are there any values of a and b for which f will be differentiable at $x = 1$?

39. For each of the following functions, do the following:

   - Use the **limit definition of the derivative** to compute the slope at the given point. No credit will be given for using any other method.

   - Use your answer for the slope to find the equation of the tangent line at the given point.

   (a) $f(x) = \sin x$ at $x = 0$.

   (b) $f(x) = \dfrac{1-x}{x}$ at $x = 1$.

40. Consider the function $f(x) = \dfrac{1}{\sqrt{x+1}}$

   (a) Find $f'(x)$ using the formal definition of the derivative.

   (b) Find x so that the tangent line to f at x has slope $-\frac{1}{16}$.

41. One day, a hitchhiker is wandering along a highway near a gas station. Suppose his position at time t is $f(t) = 360t - 3t^2$, where t is measured in minutes after noon and the position is given in feet east of the gas station. Here is the graph of $f(t)$:
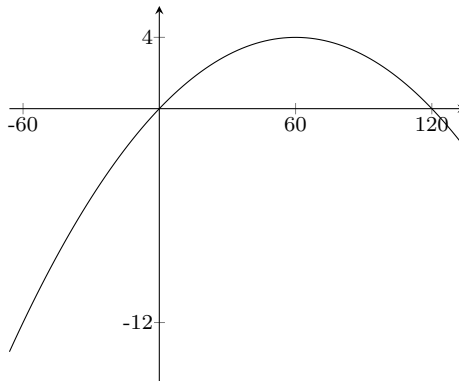
   (a) On the graph above, sketch slopes that represent the hitchhiker's instantaneous velocity at 11 am, 12:30 pm, and 2 pm. Which of these velocities is the biggest?

   (b) Were there any times when the hitchhiker's instantaneous velocity was 0? If so, when?

   (c) At what times was the hitchhiker's instantaneous velocity positive?

   (d) Sketch a rough graph of the hitchhiker's velocity as a function of time.

   (e) Using the definition of the derivative, calculate the hitchhiker's velocity at time t. This is denoted by $f'(t)$ or $\dfrac{df}{dt}$. Does this agree with your sketch?

   (f) What is the hitchhiker's acceleration at time t? Sketch a graph of his acceleration as a function of time.

42. Sara, the owner of a cupcake truck, has been experimenting with the price of the cupcakes she sells. Unsurprisingly, she has found that the number of cupcakes she sells per day depends on the price. Let $C(p)$ be the number of cupcakes she sells when the price of a cupcake is p cents.

So far, Sara has found that, when the price of a cupcake is 300 cents, she sells 600 cupcakes a day.

   (a) What are the units of $C'(300)$?

   (b) Do you expect $C'(300)$ to be positive or negative? Why?

   (c) Suppose $C'(300) = -5$. If Sara raises the price of cupcakes to 310 cents, how many cupcakes do you expect her to sell per day?

43. Let $f(x) = \left| \dfrac{x^4 - 9x^2 + 20}{x^2 - 4} \right|$. Sketch the graph of f, and then sketch the graph of f'.

44. Gromit has been growing a giant squash for Tottington Hall's annual Giant Vegetable Competition. He has carefully tracked his squash's length and weight. Let $w(\ell)$ be the squash's weight in kg when its length is ℓ cm.

   (a) What are the units of $w'(\ell)$?

   (b) Another notation for $w'(\ell)$ is $\dfrac{dw}{d\ell}$. Do you expect $\dfrac{dw}{d\ell}$ to be positive or negative? Why?

   (c) Suppose $w'(70) = 8$. (This statement can also be written as $\left.\dfrac{dw}{d\ell}\right|_{\ell=70} = 8$.) Which is the following is the most reasonable conclusion?

       A. It takes 70 days for the squash to grow to be 8 kg.

       B. When the squash is 70 cm long, it weighs 8 kg.

       C. When the squash is 70 cm longer than its current length, it will weigh 8 kg more than it currently does.

       D. When the squash grows from 70 cm to 71 cm, it will gain about 8 kg in weight.

   (d) Interpret the statement $w'(105) = 10$ in words.

45. Evaluate the following indefinite integrals.

   (a) $\displaystyle \int \left(3e^u - u^2 + 5\right) du$ (c) $\displaystyle \int \left(e^\pi + ey + \pi^y\right) dy$ (e) $\displaystyle \int \left(\dfrac{e}{x^2} + 2^x \cdot 3^x\right) dx$

   (b) $\displaystyle \int \dfrac{5}{7x}\, dx$ (d) $\displaystyle \int (\pi + x)\sqrt{x}\, dx$

46. (a) Use Newton's Method to estimate a solution to $\sin(x) = x^2 - 1$. Start with $x_0 = 0$ and calculate x_1 and x_2. *Note:* One of these you can compute by hand, for the other you'll need a calculator.

    (b) What would have happened if we'd started with $x_0 = 1$ instead of $x_0 = -1$?

47. Consider the function $f(x) = -3x^3 + 5x^2 - 1$.

(a) Are you able to algebraically solve $f(x) = 0$?

(b) Does f have a root in the interval $[0,1]$? How can you tell? How many roots are there on the interval $[0,1]$?

(c) Here is a graph of $f(x)$:
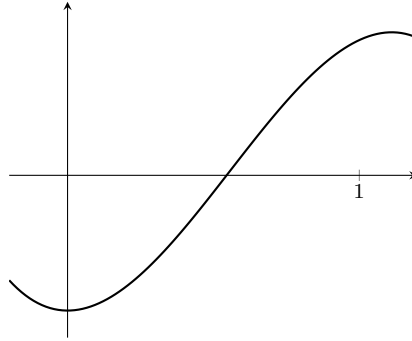
From the graph, in order to estimate the root, it seems reasonable to start with $x_0 = 1$. Use the formula for Newton's method to find x_1 and x_2. Represent your work on the graph above!

(d) Does your work in (c) mean that you can not use Newton's Method here to approximate this root?

48. True/False

(a) Let $f(x)$ be continuous on the interval $[-1,3]$. If $f(-1) = 2$ and $f(3) = 8$, then by the Intermediate Value Theorem, $f(x)$ cannot have a zero on $[-1,3]$.

(b) $\arccos(\cos(-\frac{\pi}{8})) = \frac{\pi}{8}$.

(c) Let f and g be two functions defined on $(-\infty, \infty)$, $f(0) \neq g(0)$, then we must have $\lim\limits_{x \to 0} f(x) \neq \lim\limits_{x \to 0} g(x)$.

(d) If f is even and $\lim\limits_{x \to 0^+} f(x)$ exists, then $\lim\limits_{x \to 0} f(x)$ exists.

(e) If $g(x)$ is an odd function and has a local maximum at $x = c$, then it must have a local minimum at $x = -c$.

(f) If $f(x)$ is concave down and $f'(a) = 0$, then f has a maximum $x = a$.

(g) If $f'(a) = 0$, then $f(x)$ has either a local maximum or a local minimum at $x = a$.

(h) If c is a local minimum of f, then $f''(c) > 0$.

(i) A function may have three different horizontal asymptotes.

(j) The function $|x - 2|$ is differentiable at $x = 2$.

(k) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x)$.

(l) If f is differentiable at $x = a$, then f is continuous at $x = a$.

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(m) If f is continuous at $x = a$, then f is differentiable at $x = a$.

(n) There is a function f so that $f(x) > 0$ for all x but $f'(x) < 0$ for all x.