Ceres [20 pts]

The minor planet Ceres has a mass of $M = 9 \times 10^{20}$ kg and a radius of 473 km. The Tycho Manufacturing company is working to create sustainable colonies in the Asteroid Belt, and Ceres is among their first locations. The first steps in the work involve sending raw materials from Ceres to the Anderson Space Station, located one million kilometers away, for processing into usable goods. In this problem, you can ignore the Sun and all the other objects in the Solar System.

1. [10 pts] Tycho engineers need to launch a 200 kg package of raw materials from Ceres to Anderson Station. How fast does the package need to be launched, if it needs to have a speed of 3 m/s when it arrives at Anderson?

$$U_g = \frac{mMG}{R}.$$

$$\Delta E = \Delta U_g + \Delta K = 0.$$

$$\Delta U_g = -(200\,kg)(9e20\,kg)(6.7e{-11}\,N\cdot m^2/kg^2)\left(\frac{1}{1e9\,m} - \frac{1}{4.73e5\,m}\right)$$

$$= +2.55e7\,J.$$

$$\Delta K = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = -\Delta U_g$$

$$\Rightarrow \frac{\tfrac{1}{2}mv_f^2 + \Delta U_g}{\tfrac{1}{2}m} = \frac{\tfrac{1}{2}mv_i^2}{\tfrac{1}{2}m}$$

$$\Rightarrow v_i = \sqrt{v_f^2 + \tfrac{2}{m}\Delta U_g}$$

$$= \sqrt{(3\,m/s)^2 + \frac{2}{200\,kg}(2.55e7\,J)}$$

$$= \boxed{505\,m/s}$$

2. [10 pts] To send the package, Tycho engineers built a **horizontal** spring-loaded launching mechanism on Ceres. When the apparatus is ready to engage, the package is at rest and the spring is compressed a distance of 1.5 m. What is the stiffness of the spring?

The spring needs to have properties such that upon release, the package is traveling at $v = 505$ m/s.

$$\Delta E = \Delta U_s + \Delta K + \underbrace{\cancel{\Delta U_g}}_{0 \text{ because horizontal launch}} = 0$$

$$\Rightarrow \underbrace{\cancel{\tfrac{1}{2}k_s s_f^2}}_{0 \text{ (at equilibrium)}} - \tfrac{1}{2}k s_i^2 + \tfrac{1}{2}mv_f^2 - \underbrace{\cancel{\tfrac{1}{2}mv_i^2}}_{0} = 0.$$

$v \equiv v_i$ from part 1

$$\Rightarrow -\tfrac{1}{2}k s_i^2 + \tfrac{1}{2}mv^2 = 0$$

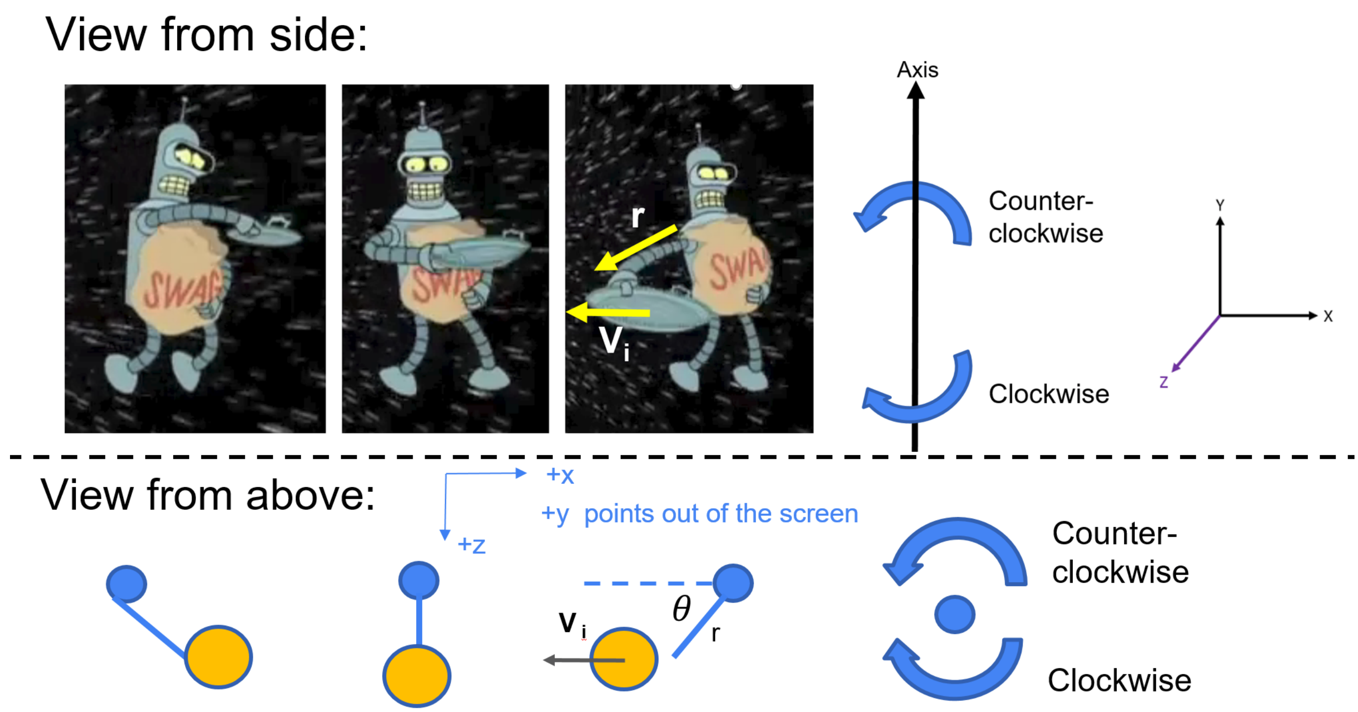
$$\Rightarrow \tfrac{1}{2}ks^2 = \tfrac{1}{2}mv^2$$

$$\Rightarrow k = \frac{mv^2}{s^2} = \frac{(200\,\text{kg})(505\,\text{m/s})^2}{(1.5\,\text{m})^2}$$

$$= \boxed{2.27e7 \text{ N/m}}$$

Bender [20 pts]

Bender is in space, far away from anything, and carrying a bag of swag. He tosses a plate of mass $m = 2$ kg with his right arm, and when the plate leaves his hand there's an angle $\theta = 30$ degrees as seen in the figure.



1. [5 pts] Which direction will Bender rotate after tossing the plate, clockwise or counterclockwise? Explain using the angular momentum principle and the right hand rule.

$\vec{L}_i = \vec{L}_f$. Initially the robot and plate are motionless, so $\vec{L}_i = 0$. Immediately after the throw, the plate has translational angular momentum in the clockwise direction ($\vec{L}$ into the page), so Bender must have angular momentum that rotates him counterclockwise ($\vec{L}$ out of the page in the above view) so that $\vec{L}_f = 0$.

2. [10 pts] If the speed of the plate is 5 m/s when Bender tosses it, and Bender's fully-extended arm is one meter long, what would be Bender's final angular momentum? Note that Bender's moment of inertia is 150 kg m^2, and that there's an angle involved (see bottom part of the figure).

$$\vec{L}_i = \vec{L}_f \quad \text{and} \quad \vec{p}_i = \vec{p}_f$$

because $\vec{\tau} = 0$

$$\vec{L}_f = \vec{L}_{rot,B} + \cancelto{0}{\vec{L}_{trans,B}} + \vec{L}_{trans,P} = 0$$

$$\Rightarrow \vec{L}_{rot,B} = -\vec{r}_P \times \vec{p}_P = -|r||p|\sin\theta(-\hat{y})$$

$$= rmv\sin\theta\,\hat{y}$$

$$= (1\,m)(2\,kg)(5\,m/s)\sin(30°)\,\hat{y}$$

$$= \boxed{5\ kg \cdot \frac{m^2}{s}\,\hat{y}}$$

3. [5 pts] Bender's getting dizzy and wants to stop spinning now. The remaining swag in the bag has a mass of 4 kg. How fast and in which direction should he toss the remaining swag so that he stops rotating altogether?

The system is now Bender + swag. $m_s = 4\,kg$

$$\vec{L}_i = \vec{L}_f$$

$$\Rightarrow \vec{L}_i = \vec{L}_{rot,B+swag} = 5\ kg \cdot \frac{m^2}{s}\,\hat{y}$$

0, treat swag as point mass

$$\vec{L}_f = \cancelto{0}{\vec{L}_{rot,B}} + \cancelto{0}{\vec{L}_{trans,B}} + \vec{L}_{trans,s} + \cancel{\vec{L}_{rot,s}}$$

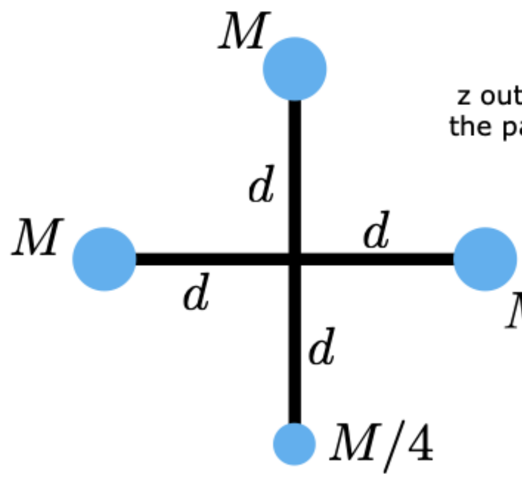
$$\Rightarrow \vec{L}_{trans,s} = \vec{L}_i$$

$\vec{L}_{trans,s} = \vec{r}_s \times \vec{p}_s = |r_s||m_s v_s|\sin\theta\,\hat{y}$ Assuming Bender throws the swag at a right angle to his arm, $|r_s||m_s v_s|\cancelto{1}{\sin(90°)}\,\hat{y} = \vec{L}_i$

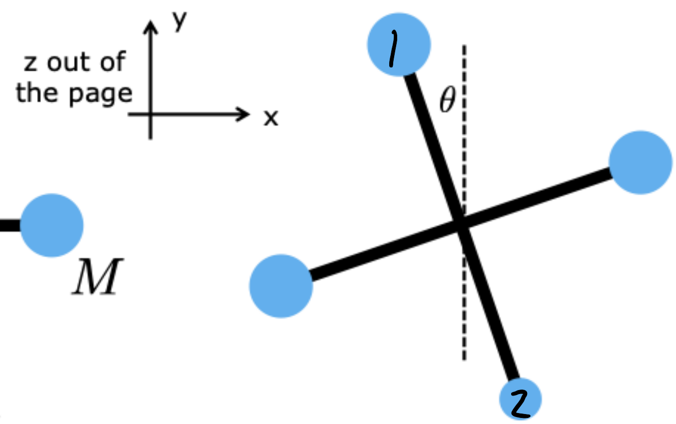
$$\Rightarrow v_s = \frac{|\vec{L}_i|}{r_s m_s} = \frac{5\ kg \cdot \frac{m^2}{s}}{(1\,m)(4\,kg)} = \boxed{1.25\,\frac{m}{s} \text{ to the right}}$$

Rods [20 pts]

A system is composed of four identical massless rods, each with length d and four balls attached to the ends of each rod. Three of the balls have mass M and the fourth one has mass $M/4$. The system is free to rotate about its center, where the bars come together. **You can think of the balls as point masses.** Earth's gravity points in the usual direction ($-\hat{y}$).



(Figure 1)



(Figure 2)

1. [5 pts] What is the net torque on the system as shown in Figure 1, where the rods are oriented along the x and y axes?

Net torque is zero; horizontal bar's torques exactly cancel.

2. [5 pts] You hold the system at rest after rotating it by a small angle θ counterclockwise (as shown in Figure 2). Then you let go. What will happen to the system now? Use the angular momentum principle to justify your answer.

The balls on the horizontal bar continue to cancel, and can be ignored. The top ball's torque is greater than the bottom ball's torque, so it will rotate counterclockwise.

3. [10 pts] Determine the final angular momentum of the system T seconds after letting it go from the initial configuration in Figure 2.

$$\vec{\tau} = \frac{\Delta \vec{L}}{\Delta t} \Rightarrow \vec{L}_f = \vec{L}_i + \vec{\tau}\Delta t$$

$$\vec{\tau}_1 = Mgd\sin\theta\,\hat{z} \,,\quad \vec{\tau}_2 = \frac{Mg}{4} d\sin\theta(-\hat{z})$$

$$\Rightarrow \vec{L}_f = \cancel{\vec{L}_i}^{\,0} + (\vec{\tau}_1 + \vec{\tau}_2)\Delta t$$

$$= \left(Mgd\sin\theta\,\hat{z} - \frac{Mg}{4} d\sin\theta\,\hat{z}\right)T$$

$$= \boxed{\frac{3}{4} MgdT\sin\theta\,\hat{z}}$$

EXTRA CREDIT [5 pts]

1. [1 pt] Write the equation for the Momentum Principle. To earn credit your equation must be fully correct.

$$\vec{F}_{net} = \frac{\Delta \vec{p}}{\Delta t}$$

2. [1 pt] Write the equation for the Energy Principle. To earn credit your equation must be fully correct.

$$\Delta E = W_{ext} + Q$$

3. [1 pt] Write the equation for the Angular Momentum Principle. To earn credit your question must be fully correct.

$$\vec{\tau}_{net} = \frac{\Delta \vec{L}}{\Delta t}$$

4. [2 pts] The Lyman Alpha line appears in the emission line spectrum of hydrogen when the electron goes from the $N = 2$ to the $N = 1$ level, emitting a photon in the process. What is the energy of the Lyman Alpha photon?

- 0 eV
- 3.4 eV
- 6.8 eV
- 10.2 eV
- 13.6 eV

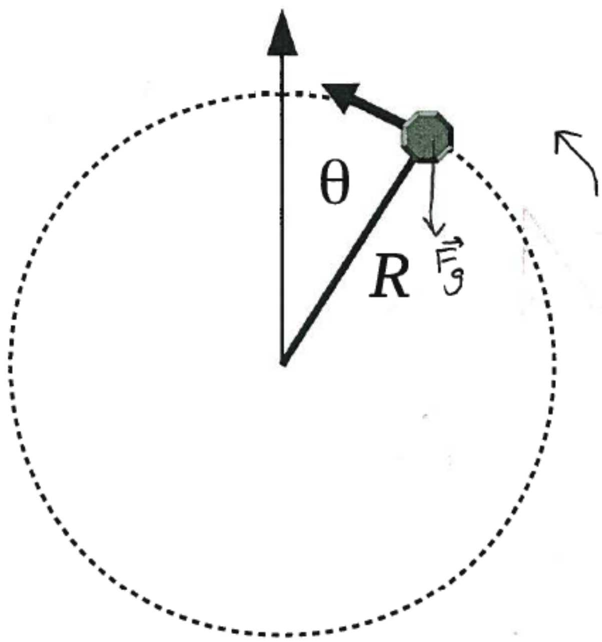
$$\Delta E = -13.6\,\text{eV}\left(\frac{1}{2^2} - \frac{1}{1^2}\right)$$

PHYS 2211 Modern Final Review Worksheet – Reading Day Fall 2021
Worked Solutions

Curving Motion / Gravitation

Problem 1

On the Earth, a rock of mass m is tied to the end of a rope of length R. The rock is swung counterclockwise in a circle of radius R in a vertical plane (gravity points down). Consider the rock when the rope makes an angle of θ with the vertical, as shown in the diagram. At this location, the tension in the rope is a known quantity T and the speed of the rock is decreasing.



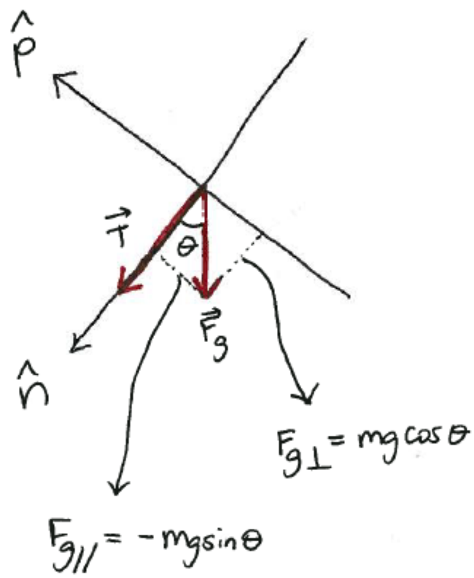
(a 5pts) Is the perpendicular component of the net change in momentum for the rock zero? Explain briefly how you know this.

3pts $\left\{ \left(\frac{d\vec{p}}{dt}\right)_\perp \neq 0 \right.$ because the rock is turning (curving trajectory) $\}$ 2pts

Also: $\left(\frac{dp}{dt}\right)_\perp = \frac{mv^2}{R}$, and $m \neq 0$, $v^2 \neq 0$, $R \neq 0$

(b 5pts) Is the parallel component of the net change in momentum for the rock zero? Explain briefly how you know this.

3pts $\left\{ \left(\frac{d\vec{p}}{dt}\right)_\parallel \neq 0 \right.$ because the speed of the rock is changing (decreasing at this location) $\}$ 2pts (c 7pts) Determine the parallel component of the net force on the rock in terms of the known quantities given in the problem statement.



$$\vec{F}_{net\,\parallel} = \vec{F}_{g\parallel} + \underset{0}{\cancel{\vec{T}_{\parallel}}} =$$

$$= \vec{F}_{g\parallel} =$$

$$= \boxed{-mg\sin\theta\ \hat{p}}$$

$$\begin{bmatrix} -0.5 \\ -1.0 \\ -2.5 \\ -6.5 \end{bmatrix}$$

(d 8pts) Calculate the speed of the rock in terms of the known quantities given in the problem statement.

$$\left(\vec{F}_{net}\right)_\perp = \left(\frac{d\vec{p}}{dt}\right)_\perp$$

$$\vec{F}_{g\perp} + \vec{T}_\perp = \frac{mv^2}{R}\hat{n}$$

$$mg\cos\theta\,\hat{n} + T\hat{n} = \frac{mv^2}{R}\hat{n}$$

$$\frac{R}{m}\left(mg\cos\theta + T\right) = v^2$$

$$\boxed{v = \sqrt{Rg\cos\theta + \frac{RT}{m}}}$$

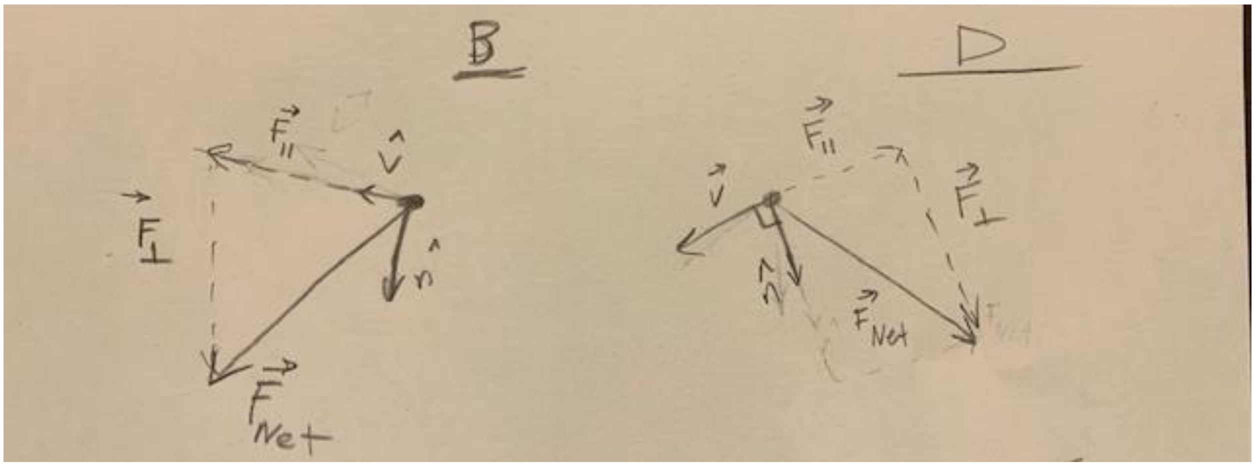
$$\begin{bmatrix} -0.5 \\ -1.0 \\ -2.5 \\ -6.5 \end{bmatrix}$$

Problem 2

a) A and C. At these points, F_{Net} and $F\perp$ are in the same direction. So no parallel component.

b) None. There will always be force perpendicular to the motion because the motion is constantly curving.

c) D. We can rule out A and C because it will be 0 at those points. FBD below shows $F||$ is opposite the velocity at point D and not point B.



d) None. The gravitational force never points away from the center of curvature.

e) C. Minimum radius of curvature occurs at this point, meaning greatest velocity.

Greatest Speed Occurs at point C.

$$R = \frac{b^2}{a}$$

$$\vec{F}_{Net} = \vec{F}_{\perp} = \vec{F}_g = \frac{-GmM}{|\vec{r}|^2}\hat{r} = \frac{m|V|^2}{R}\hat{n}$$

$|\vec{r}| = b$ $\quad$ $\hat{r} = -\hat{n}$ @ C

$$\frac{GM}{b^2} = \frac{|V|^2}{\frac{b^2}{a}} \Rightarrow |V|^2 = GMa$$

$$|V_{max}| = \sqrt{GMa}$$

Lowest Speed Occurs at point A

$$R = \frac{a^2}{b}$$

$$\frac{-GMm}{|\vec{r}|^2}\hat{r} = \frac{m|V|^2}{R}\hat{n} \Rightarrow \frac{GM}{a^2} = \frac{|V|^2}{\frac{a^2}{b}}$$

$$|V_{min}| = \sqrt{GMb} \qquad \frac{V_{max}}{V_{min}} = \frac{\sqrt{GMa}}{\sqrt{GMb}} = \sqrt{\frac{a}{b}}$$

f)

Energy principle

Problem 3

(a 10pts) A bowling ball of mass m is gently placed onto the un-stretched trampoline. After waiting a few seconds the bowling ball comes to rest and the trampoline is stretched an amount d_{static}. Taking the bowling ball as your system, how much work did the trampoline do as the ball traveled from $d = 0$ to $d = d_{static}$?

$$W_t = -\Delta U_t$$

$$U_{t,f} = \frac{1}{4}kd^4 \qquad U_{t,i} = 0$$

$$W_t = -(U_{t,f} - U_{t,i})$$

$$= -U_{t,f} + U_{t,i}$$

$$= -\frac{1}{4}kd^4 + 0$$

$$\boxed{W_t = -\frac{1}{4}kd^4}$$

Problem 3

Part (A) Alternative method!

$\uparrow +y$

$\vec{r}_0 = 0$

$\vec{r}_1 = -d_{static}$

Ball moves downward

$$W_{tramp} = \int_{\vec{r}_0}^{\vec{r}_1} \vec{F}_{tramp} \cdot d\vec{r}$$

$\downarrow d\vec{r}$ $\uparrow F$

$\theta = 180^\circ$

$-d = y \Rightarrow |F| = ky^3$ from problem statement

$$W_{tramp} = \int_0^{-d_{static}} |\vec{F}||d\vec{r}| \cos\theta_F$$

$$= \int_0^{-d_{static}} -ky^3\, dy$$

$$= -\frac{1}{4}ky^4 \Big|_0^{-d_{static}}$$

$$\boxed{W_{tramp} = -\frac{1}{4}k\left(d_{static}\right)^4}$$

Part (b)

system: trampoline and bowling Ball and Earth

External Forces = N/A

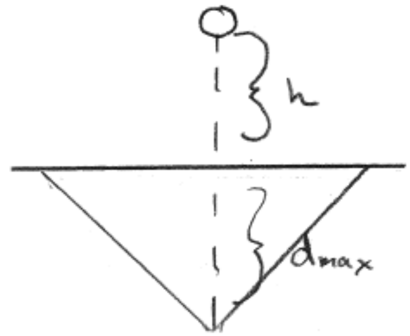
Net Work = 0

$$\cancel{W}^{0} + \cancel{Q}^{0} = \Delta E = \cancel{\Delta K}^{0} + \Delta U_g + \Delta U_{tramp}$$

$$\Rightarrow \Delta U_g = \Delta U_{tramp}$$

(b 15pts) In a different experiment, a bowling ball of mass m is dropped from rest a height h above the un-stretched trampoline (d=0). Determine the maximum stretch d_{max} of the trampoline during the bounce of the bowling ball.

$$\cancel{W} + \Delta U_g + \Delta U_t = 0$$



$$\Delta U_g = -\Delta U_t$$

$$U_{g,i} = mg(h + d_{max})$$

$$U_{g,f} = 0$$

$$\Delta U_g = -mg(h + d_{max})$$

$$U_{t,i} = 0$$

$$U_{t,f} = \frac{1}{4} k\, d_{max}^4$$

$$\Delta U_t = \frac{1}{4} k\, d_{max}^4$$

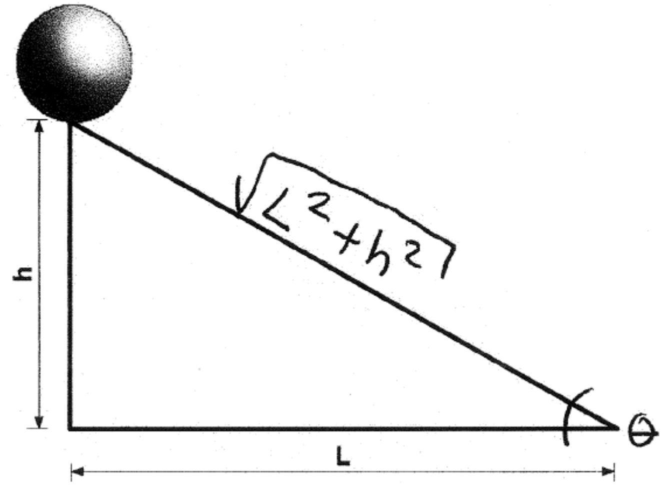
$$\Rightarrow \frac{1}{4} k d_{max}^4 = + mg(h + d_{max})$$

$$\frac{1}{4} k d_{max}^4 = + mgh + mg d_{max}$$

$$\boxed{d_{max}^4 = \frac{4mgh}{k} + \frac{4mg}{k} d_{max}}$$

$$\boxed{\begin{matrix} -1 \\ -2 \\ -4.5 \\ -12 \end{matrix}}$$

Problem 4



(a 5pts) Choose your system to be the sphere only. As the sphere rolls down the hill, determine the work done on the real system.

$$W = \vec{F} \cdot d\vec{r}$$

3pts $\left\{ \begin{array}{l} \text{The only force that does work here is gravity.} \\ \text{Friction is present but does no work on the real system.} \end{array} \right.$

So: $W = W_g = Mgh$

because gravity acts along the vertical displacement

☆ $\boxed{W = Mgh}$ ☆ 2pts

(b 10pts) The sphere starts at the top of the hill from rest and rolls to the bottom. Determine the speed of the sphere when it reaches the bottom. Apply the energy principle on the real system consisting only of the sphere.

$$\Delta K_{trans} + \Delta K_{rot} = W$$

$$\frac{1}{2} M (v_f^2 - \cancel{v_i^2}) + \frac{1}{2} I (\omega_f^2 - \cancel{\omega_i^2}) = Mgh$$

$$\frac{1}{2} M v_f^2 + \frac{1}{2} I \omega_f^2 = Mgh$$

We know that $\omega = v/R$, so we have...

$$\frac{1}{2} M v_f^2 + \frac{1}{2} I \frac{v_f^2}{R^2} = Mgh$$

$$M v_f^2 + I \frac{v_f^2}{R^2} = 2Mgh$$

$$v_f^2 + \frac{I}{MR^2} v_f^2 = 2gh$$

$$\left(1 + \frac{I}{MR^2}\right) v_f^2 = 2gh$$

$$v_f^2 = \frac{2gh}{1 + \frac{I}{MR^2}}$$

−0.5
−1.5
−3.0
−8.0

For a solid sphere, $I = \frac{2}{5} MR^2$, so we get...

$$v_f^2 = \frac{2gh}{1 + \frac{2\cancel{M}\cancel{R^2}}{5\cancel{M}\cancel{R^2}}} = \frac{2gh}{1 + \frac{2}{5}} = \frac{2gh}{7/5} = \frac{5}{7}(2gh)$$

$$v_f^2 = \frac{10gh}{7}$$

$$\boxed{v_f = \sqrt{\frac{10gh}{7}}}$$

(c 5pts) Determine the average frictional force acting on the sphere as it rolled to the bottom of the hill. Apply the energy principle on the point particle system consisting only of the sphere. You should use your result from part (a).

(2pts) $\Delta K_{trans} = W_g + W_{friction}$

$$\frac{1}{2} M v_f^2 = Mgh + \vec{f} \cdot d\vec{r}$$

friction acts along the displacement down the hill, in the direction opposite of movement (2pts)

$$\frac{1}{2} M v_f^2 - Mgh = -f\sqrt{L^2+h^2}$$

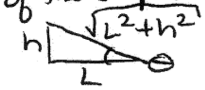
$$\frac{1}{\cancel{2}} M \left(\frac{\cancel{10}^{5} gh}{7}\right) - Mgh = -f\sqrt{L^2+h^2}$$

$$\frac{5Mgh}{7} - Mgh = -f\sqrt{L^2+h^2}$$

$$-\frac{2Mgh}{7} = -f\sqrt{L^2+h^2}$$

$$☆ \boxed{f = \frac{2Mgh}{7\sqrt{L^2+h^2}}} ☆ \text{(1pt)}$$

Note also, if θ is the angle of the slope/hill:



then $\frac{h}{\sqrt{L^2+h^2}} = \cos\theta$, and the friction can be expressed as:

$$☆ \boxed{f = \frac{2}{7} Mg\cos\theta} ☆$$

(d 5pts) A second sphere with the same mass total M and radius R but with a hollow core is rolled down the hill. This hollow sphere will have a higher density then the solid sphere so that they have the same mass. Does it have a final speed (at the bottom) that is greater than, equal to or less than the solid sphere from part (a)? Briefly explain how you know this.

The solid sphere has its mass distributed all throughout its volume. In contrast, the hollow sphere has all its mass at the surface/outer edge (distance R from center). This means that: $\boxed{I_{hollow} > I_{solid}}$ → (3pts)

In Part (b) we saw that the final speed of the sphere depends on I in this way:

$$v_f^2 = \frac{2gh}{1 + \frac{I}{MR^2}}$$

When I is bigger, the denominator is bigger, and therefore V_f becomes smaller. Thus, the hollow sphere will move slower than the solid sphere as it rolls down the hill.

$$☆ \boxed{V_{hollow} < V_{solid}} ☆ \text{(2pts)}$$

Energy graphs

Problem 5

(a 10pts) The block continues downward. When the bottom of the block is 0.3 m above the floor, what is its speed? (Consider the block, the spring and earth as the system).

Initial state: block @ 0.8 m above floor (no contact with spring)

Final state: block @ $L = 0.3$ m above floor (in contact with spring)

$$\Delta E = 0 \Rightarrow \Delta K_T + \Delta U_T = 0$$

$$\Delta K_{block} + \cancel{\Delta K_{spring}} + \cancel{\Delta K_{Earth}} + \Delta U_{block-Earth} + \Delta U_{spring} = 0$$

$$(K_{bf} - K_{bi}) + (U_{bf} - U_{bi}) + (U_{sf} - \cancel{U_{si}}) = 0$$

$$\frac{1}{2}m(v_f^2 - v_i^2) + mg(L-h) + \frac{1}{2}k(L-L_0)^2 = 0$$

$$\frac{1}{2}(3)(v_f^2 - 2^2) + (3)(9.81)(0.3 - 0.8) + \frac{1}{2}(2000)(0.3-0.4)^2 = 0$$

$$\frac{3}{2}v_f^2 - 6 - 14.715 + 10 = 0$$

$$\frac{3}{2}v_f^2 - 10.715 = 0$$

$$\frac{3}{2}v_f^2 = 10.715$$

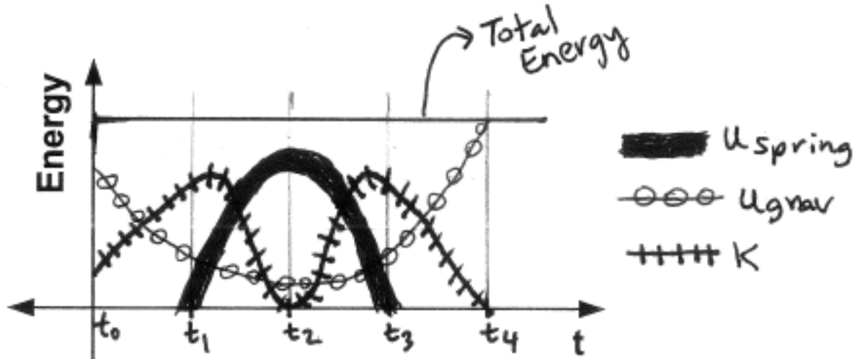
$$v_f^2 = \left(\frac{2}{3}\right)(10.715)$$

$$v_f^2 = 7.143$$

$$\boxed{v_f = 2.673 \text{ m/s}}$$

$$\boxed{\begin{matrix} -0.5 \\ -1.5 \\ -3.0 \\ -8.0 \end{matrix}}$$

(c 5pts) Sketch a graph of spring potential energy, gravitational potential energy, and the kinetic energy as a function of time.



1pt each

t_0 = problem statement
t_1 = block comes in contact with spring
t_2 = spring is maximally compressed
t_3 = block loses contact with spring
t_4 = block is at maximum height

Problem 6

(a 5pts) Determine the velocity of the rock the instant it reaches the surface of the Earth. Your answer should not be numeric.

3pt → $\Delta E = \Delta K + \Delta U = 0$

$$K_f - \cancel{K_i}^{\,0} + U_f - \cancel{U_i}^{\,0} = 0$$

$$K_f + U_f = 0$$

$$\frac{1}{2}mv^2 + \frac{-GMm}{r_f} = 0$$

$$\frac{1}{2}\cancel{m}v^2 = \frac{GM\cancel{m}}{1}$$

$$\frac{1}{2}v^2 = \frac{GM}{R}$$

$$v^2 = \frac{2GM}{R}$$

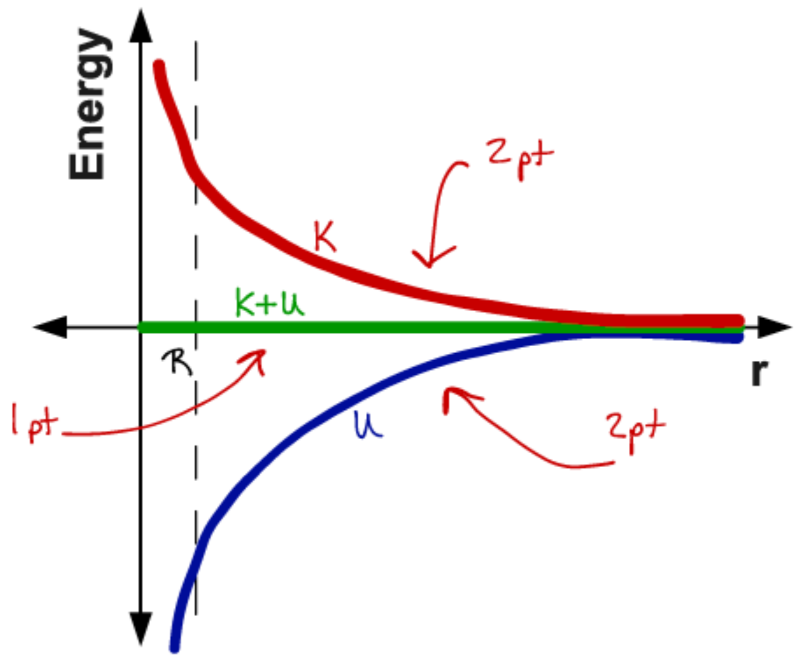
2pt

$$\boxed{v = \sqrt{\frac{2GM}{R}}}$$

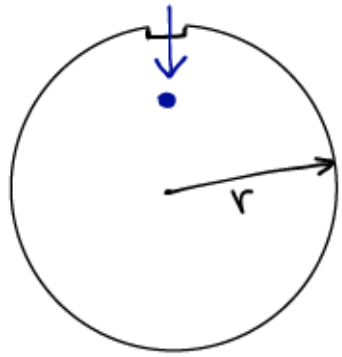
direction: towards center of Earth

where: $\begin{cases} M = \text{mass of Earth} \\ R = \text{radius of Earth} \end{cases}$

(b 5pts) For the case considered above, sketch the: Kinetic, Potential, and Total energy for the Earth+Rock system.



(c 10pts) Determine the velocity of the rock the instant it reaches the center of the Earth. Your answer should not be numeric.



✓ System: only the rock

✓ Initial: rock @ surface $(r = R)$

✓ Final: rock @ center $(r = 0)$

✓ $W = \int \vec{F} \cdot d\vec{r} = \int_R^0 \frac{-mgr}{R} dr = \frac{-mg}{R} \int_R^0 r\, dr =$

$= \frac{-mg}{R} \left[\frac{r^2}{2} \right]_R^0 = \frac{+mg}{\cancel{R}} \left(\frac{+R^{\cancel{2}}}{2} \right) = \frac{mgR}{2}$

✓ $\Delta E = \Delta K = W$

$K_f - \cancel{K_i}^{\,0} = W$

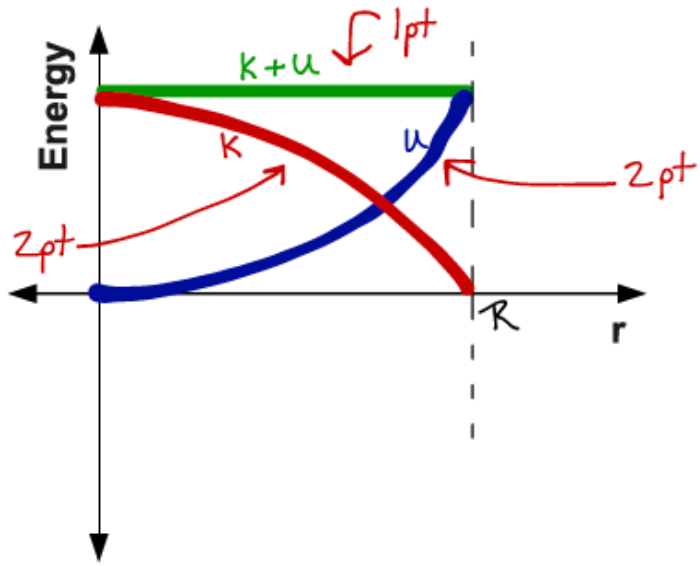
$\frac{1}{\cancel{2}} \cancel{m} v^2 = \frac{\cancel{m} gR}{\cancel{2}}$

$v^2 = gR$

$\boxed{v = \sqrt{gR}}$ → direction: away from the center of the Earth

$\boxed{\begin{matrix} -0.5 \\ -1.5 \\ -3.0 \\ -8.0 \end{matrix}}$

(d 5pts) For the case of the rock falling through the Earth, the potential energy of the Earth+Rock system is given by $mgr^2/(2R)$. On the graph below, sketch the Kinetic, Potential and Total energy for the Earth+Rock system.



All (Extra credit 5pts) How does your answer to part (c) compare to your answer in part (a)? Hint: the ratio of the two velocities should not depend on any of the parameters in the problem.

✓ What is g? $\Rightarrow \cancel{m}g = \frac{GM\cancel{m}}{R^2} \Rightarrow g = \frac{GM}{R^2}$

✓ Part A: $V_A = \sqrt{\frac{2GM}{R}}$

✓ Part C: $V_c = \sqrt{gR} = \sqrt{\frac{GM}{R^{\cancel{2}}}\cancel{R}} = \sqrt{\frac{GM}{R}}$

$\Rightarrow$ Ratio: $\frac{V_c}{V_A} = \frac{\sqrt{\cancel{GM}/\cancel{R}}}{\sqrt{2\cancel{GM}/\cancel{R}}} = \boxed{\frac{1}{\sqrt{2}}}$ or $\frac{V_A}{V_c} = \sqrt{2}$

* The speed at crashing (Part A) is $\sqrt{2}$ times larger than the speed at the center of Earth (Part C).

Angular momentum and Torque for solid extended system

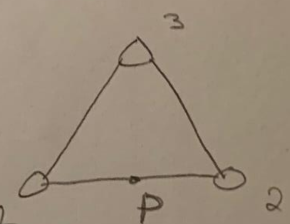
Problem 7

Problem 7 — 7/27

(a) Moment of Inertia =

$I = \sum_{i=0} m_i r_i^2$ where r is the distance to from the axis of rotation to the cm of the object.

$m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$



$m_1 = 2M$ $\quad m_3 = M$

$r_1 = a$ $\quad r_3 = \sqrt{b^2 - a^2}$

$m_2 = 2M$

$r_2 = a$

$(2M)(a)^2 + (2M)(a)^2 + M\left(\sqrt{b^2-a^2}\right)^2$

$4Ma^2 + Mb^2 - Ma^2$

$\boxed{I = 3Ma^2 + Mb^2}$

(b) For rigid Body, $\vec{L} = I\omega$

By Angular momentum Principle

$$\vec{L}_f = \vec{L}_o + \Sigma \tau \Delta t$$

$$I\vec{\omega}_f = \cancel{I\vec{\omega}_i}^{\,0} + \tau_{app}\,\Delta t$$

$$\boxed{\tau_{app} = \frac{I\omega}{\Delta t} = \frac{(3Ma^2 + Mb^2)\,\omega}{\Delta t}}$$

Problem 8

Method #1

$$I = Mr^2 = 2mb^2$$

$\vec{L}_{trans,B} = I\vec{\omega} = 2mb^2\omega$ (direction: into the page)

$$\Rightarrow \boxed{\vec{L}_{trans,B} = -2mb^2\omega\,\hat{z}}$$

−0.5
−1.5
−3.0
−8.0

(either method)

Method #2

$$\vec{L}_{trans,B} = \vec{r}_{cm} \times M_{sys}\vec{V}_{cm}$$
$$= r_{cm} M_{sys} V_{cm}\,(-\hat{z})$$

always 90° and by right-hand-rule, $\vec{r} \times \vec{v}$ points into the page $(-\hat{z})$

✓ $M_{sys} = 2m$

✓ $V_{cm} = \omega r_{cm}$

$$\Rightarrow \vec{L}_{trans,B} = (r_{cm})(2m)(\omega r_{cm})(-\hat{z}) = 2m\omega(r_{cm})^2\,(-\hat{z})$$

$$\Rightarrow \boxed{\vec{L}_{trans,B} = \langle 0, 0, -2mb^2\omega \rangle}$$

(b 5pts) If the total angular momentum for this system (about the point B) is zero, calculate the rotational angular momentum for the barbell about its center of mass. $\vec{L}_{rot,cm}$ (magnitude and direction)

$$\left.\begin{aligned} \vec{L}_{total,B} &= \vec{L}_{trans,B} + \vec{L}_{rot,CM} = 0 \\ \vec{L}_{rot,CM} &= -\vec{L}_{trans,B} \end{aligned}\right\} \text{3pts}$$

$$\vec{L}_{rot,CM} = -(-2mb^2\omega\,\hat{z})$$

$$\boxed{\vec{L}_{rot,CM} = 2mb^2\omega\,\hat{z}} \longrightarrow \text{2pts}$$

(c 5pts) Calculate the moment of inertia I for the barbell about its center of mass.

$$I_{barbell} = (2m)(d/2)^2 = (\cancel{2}m)\left(\frac{1}{\cancel{4}}d^2\right)$$

$$\boxed{I_{barbell} = \frac{1}{2}md^2}$$

All

(d 5pts) Determine the unknown angular speed of the barbell about its center of mass.

$$\left.\begin{aligned} &✓\ \vec{L}_{rot,CM} = I_{barbell}\vec{\Omega} = \frac{1}{2}md^2\vec{\Omega} = \frac{1}{2}md^2\Omega\,\hat{z} \text{ (b/c counterclockwise)} \\ &✓\ \vec{L}_{rot,CM} = 2mb^2\omega\,\hat{z} \text{ (from Part b)} \longrightarrow \text{Check for PDE} \end{aligned}\right\} \text{3pts}$$

$$\Rightarrow \frac{1}{2}\cancel{m}d^2\Omega\,\hat{z} = 2\cancel{m}b^2\omega\,\hat{z}$$

$$\boxed{\Omega = \frac{4b^2\omega}{d^2}} \longrightarrow \text{2pts}$$

Angular Momentum and Torque for a system with particles

Problem 9

(a 5pts) Determine the angular momentum of particle 1 relative to the point $\vec{A} = <0,0,0>$ m.?

$$\vec{L}_1 = \vec{r}_1 \times \vec{p}_1 = \langle 5,0,0\rangle \times 5\langle -4,-12,0\rangle =$$
$$= \langle 5,0,0\rangle \times \langle -20,-60,0\rangle = \dots$$

3pts

$$\begin{vmatrix} x & y & z \\ 5 & 0 & 0 \\ -20 & -60 & 0 \end{vmatrix}$$

x: $0(0) - 0(-60) = 0$

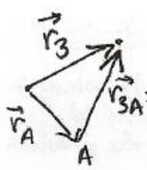
y: $0(-20) - 5(0) = 0$

z: $5(-60) - 0(-20) = -300$

2pts

$$\boxed{\vec{L}_1 = \langle 0,0,-300\rangle \ \mathrm{kg\,m^2/s}}$$

(b 5pts) Determine the angular momentum of particle 3 relative to the point $\vec{A} = <10,0,5>$ m.?



$\vec{r}_{3A} = \vec{r}_3 - \vec{r}_A$

SAME

$$\vec{L}_{3A} = \vec{r}_{3A} \times \vec{p}_3 =$$
$$= \left[\langle 0,0,5\rangle - \langle 10,0,5\rangle\right] \times 10\langle 8,0,-16\rangle =$$
$$= \langle -10,0,0\rangle \times \langle 80,0,-160\rangle = \dots$$

$$\begin{vmatrix} x & y & z \\ -10 & 0 & 0 \\ 80 & 0 & -160 \end{vmatrix}$$

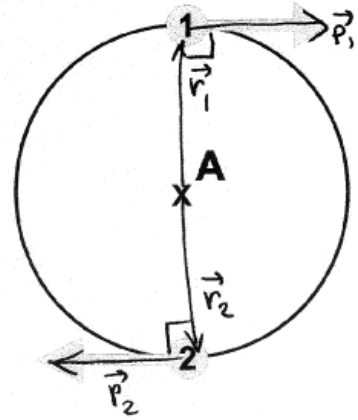
x: $0(-16) - 0(0) = 0$

y: $0(80) - (-10)(-160) = 0 - 1600 = -1600$

z: $-10(0) - 0(80) = 0$

$$\boxed{\vec{L}_{3A} = \langle 0,-1600,0\rangle \ \mathrm{kg\,m^2/s}}$$

(c 5pts) A race car drives clockwise around a circular track at a constant speed as seen in the figure. Determine the direction of the angular momentum of the car relative to the center of the track.



All

Into the page $\otimes$

(d 5pts) How does the angular moment of the car, relative to the center of the track, change as the car goes from location 1 to location 2? Your answer should be supported by physics principles.

$L = rp\sin\theta$

For both ① and ②, $\theta = 90°$, so $\sin\theta = 1$.

2pts

So, @ ①: $L_1 = r_1 p_1$

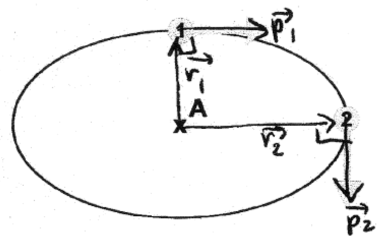
@ ②: $L_2 = r_2 p_2$

r_1 and r_2 are the same (circular track)

p_1 and p_2 are the same (constant speed)

Therefore: $L_1 = L_2$ ⇒ $\boxed{\text{No change in angular momentum}}$ ☆ 3pts

(e 5pts) A race car drives clockwise around an elliptical track at a constant speed as seen in the figure. How does the angular moment of the car, relative to the center of the track, change as the car goes from location 1 to location 2? Your answer should be supported by physics principles.



$L = rp\sin\theta$

For both ① and ②, $\theta = 90°$, so $\sin\theta = 1$.

2pts

@ ①: $L_1 = r_1 p_1$

@ ②: $L_2 = r_2 p_2$

p_1 and p_2 are the same (constant speed)

r_1 and r_2 are NOT the same.

r_1 is semiminor axis, r_2 is semimajor axis $\Rightarrow r_1 < r_2$

Since $r_1 < r_2$, then $\boxed{L_1 < L_2}$

1pt

⇒ $\boxed{\text{Angular momentum increased from point ① to point ②}}$ ☆ 2pts

Problem 10

(c 5pts) What is the angular momentum of particle 2 relative to the origin?

1pt $\vec{L} = \vec{r} \times \vec{p}$ $\quad \vec{r}_2 \times \vec{p}_2 = m(\vec{r}_2 \times \vec{V}_2) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 5 & 0 \\ -10 & 0 & -5 \end{vmatrix} \cdot 10 \ \frac{kg\,m^2}{s}$

3pts $= \left[(-25)\hat{x} + (50)\hat{z}\right] 10 \ \frac{kg\,m^2}{s}$

$= \langle -250, 0, 500 \rangle \ \frac{kg\,m^2}{s}$

1pt $\boxed{\vec{L} = \langle -250, 0, 500 \rangle \ \frac{kg\,m^2}{s}}$

(d 10pts) What is the angular momentum of particle 3 relative to the location of particle 2?

(obs) (src)

$\vec{r} = \vec{r}_3 - \vec{r}_2 = \langle 0, 0, 5 \rangle m - \langle 0, 5, 0 \rangle m$

$\vec{r} = \langle 0, -5, 5 \rangle m$

$\vec{V} = \vec{V}_3 = \langle 0, 0, -10 \rangle m/s$

−0.5
−1.5
−3.0
−8.0

$|\vec{r} \times \vec{V}| = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -5 & 5 \\ 0 & 0 & -10 \end{vmatrix} = \hat{x}(50)$

$\vec{L} = m_3 |\vec{r} \times \vec{v}| = (15) \langle 50, 0, 0 \rangle \ \frac{kg\,m^2}{s}$

$\boxed{\vec{L} = \langle 750, 0, 0 \rangle \ \frac{kg\,m^2}{s}}$

(c) $\vec{L}_{tot} = \vec{L}_{trans} + \vec{L}_{rot}$

$$= \vec{r}_{cm} \times (\vec{p}_1 + \vec{p}_2 + \vec{p}_3) + (\vec{r}_{1,cm} \times \vec{p}_1 + \vec{r}_{2,cm} \times \vec{p}_2 + \vec{r}_{3,cm} \times \vec{p}_3)$$

so we need $\vec{r}_{cm}$ vector.

$$\vec{r}_{cm} = \frac{\sum_{i=0} m_i \vec{r}_i}{\sum_{i=0} m_i} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}$$

$$\vec{r}_{cm} = \frac{\langle 25,0,0\rangle + \langle 0,50,0\rangle + \langle 0,0,75\rangle \ \text{kg}\cdot\text{m}}{5+10+15 \quad \text{kg}}$$

$$\vec{r}_{cm} = \left\langle \frac{5}{6}, \frac{5}{3}, \frac{5}{2} \right\rangle$$

$$\vec{L}_{trans} = \left\langle \frac{5}{6}, \frac{5}{3}, \frac{5}{2} \right\rangle \times \underbrace{(m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3)}$$

$$\begin{aligned} &\langle 25, -50, 75\rangle = \vec{p}_1 \\ +&\langle -100, 0, -50\rangle = \vec{p}_2 \\ +&\langle 0, 0, -150\rangle = \vec{p}_3 \\ \hline &\langle -75, -50, -125\rangle \ \text{kg}\cdot\tfrac{\text{m}}{\text{s}} \end{aligned}$$

$$\vec{L}_{trans} = \left\langle \frac{5}{6}, \frac{5}{3}, \frac{5}{2} \right\rangle \text{m} \times \langle -75, -50, -125\rangle \ \text{kg}\cdot\frac{\text{m}}{\text{s}}$$

$$\vec{L}_{trans} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{5}{6} & \frac{5}{3} & \frac{5}{2} \\ -75 & -50 & -125 \end{vmatrix} = \left(\left(\tfrac{5}{3}\cdot -125\right)-\left(\tfrac{5}{2}\cdot -50\right)\right)\hat{i} - \left(\left(\tfrac{5}{6}\cdot -125\right)-\left(\tfrac{5}{2}\cdot -75\right)\right)\hat{j} + \left(\left(\tfrac{5}{6}\cdot -50\right)-\left(\tfrac{5}{3}\cdot -75\right)\right)\hat{k}$$

$$\boxed{\vec{L}_{trans} = \left\langle -\frac{250}{3}, \frac{-250}{3}, \frac{250}{3} \right\rangle}$$

Now for $\vec{L}_{rot}$

$$\vec{r}_{1,cm} = \langle 5,0,0\rangle - \left\langle \tfrac{5}{6}, \tfrac{5}{3}, \tfrac{5}{2}\right\rangle = \left\langle \tfrac{25}{6}, \tfrac{-5}{3}, \tfrac{-5}{2}\right\rangle$$

$$\vec{r}_{2,cm} = \langle 0,5,0\rangle - \left\langle \tfrac{5}{6}, \tfrac{5}{3}, \tfrac{5}{2}\right\rangle = \left\langle \tfrac{-5}{6}, \tfrac{10}{3}, \tfrac{-5}{2}\right\rangle$$

$$\vec{r}_{3,cm} = \langle 0,0,5\rangle - \left\langle \tfrac{5}{6}, \tfrac{5}{3}, \tfrac{5}{2}\right\rangle = \left\langle \tfrac{-5}{6}, -\tfrac{5}{3}, \tfrac{5}{2}\right\rangle$$

$$\vec{r}_{1,cm} \times \vec{p}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{25}{6}, & \frac{-5}{3}, & \frac{-5}{2} \\ 25 & -50 & 75 \end{vmatrix} = \left(\left(\tfrac{-5}{3}\cdot 75\right)-\left(\tfrac{-5}{2}\cdot -50\right)\right)\hat{i} - \left(\left(\tfrac{25}{6}\cdot 75\right)-\left(\tfrac{-5}{2}\cdot 25\right)\right)\hat{j} + \left(\left(\tfrac{25}{6}\cdot -50\right)-\left(\tfrac{-5}{3}\cdot 25\right)\right)\hat{k}$$

$$\boxed{= \langle 0, -375, -166.7\rangle}$$

$$\vec{r}_{2,cm} \times \vec{p}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-5}{6} & \frac{10}{3} & \frac{-5}{2} \\ -100 & 0 & -50 \end{vmatrix} = \left(\left(\tfrac{10}{3}\cdot -50\right)-0\right)\hat{i} - \left(\left(\tfrac{-5}{6}\cdot -50\right)-\left(\tfrac{-5}{2}\cdot -100\right)\right)\hat{j} + \left((0)-\left(-100\cdot\tfrac{10}{3}\right)\right)\hat{k}$$

$$\boxed{= \langle 166.7,\ 208.3,\ 333.3 \rangle}$$

$$\vec{r}_{3,cm} \times \vec{p}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{5}{6} & -\frac{5}{3} & \frac{5}{2} \\ 0 & 0 & -150 \end{vmatrix} = \begin{matrix} ((-\frac{5}{3} \cdot -150) - 0)\ \hat{i} \\ -((-\frac{5}{6} \cdot -150) - 0)\ \hat{j} \\ +(0 - 0)\ \hat{k} \end{matrix}$$

$$\boxed{= \langle 250, -125, 0 \rangle}$$

Now the total $\vec{L}$ can be found by summing all boxed values.

$$\boxed{\vec{L}_{tot} = \left\langle \frac{1000}{3}, -375, 250 \right\rangle \ \mathrm{kg} \cdot \frac{m^2}{s}}$$

*A lot of calculations here... I may have made an error somewhere along the way. The important part is the process!

Additional Practice

Problem 11

Check each choice Individually

A) object two has same Mass distributed farther from center of Mass, so $I_2 > I_1$ ✓

B) Object two has much less Mass. In fact, object 1 contains object two but has additional mass b/w center and surface. Adding Mass increases I so $I_1 > I_2$ ✗

C) Similar logic to ~~chioce~~ choice b. Object 2 more massive, same geometry so, $I_2 > I_1$ ✓

D) object 1 object 2

The mass is clearly distributed further from the axis in object 2, so $I_2 > I_1$ ✓

E) By the parallel axis theorem,

$I = I_o + Md^2$, where I_o is moment about center of mass.

Therefore, an axis passing through center of mass is minimum for all parallel axes, regardless of object geometry. So $I_2 < I_1$ ✗

Answer:

A, C, D $\quad I_2 > I_1$

Problem 12

Problem 12

Conservation of linear momentum

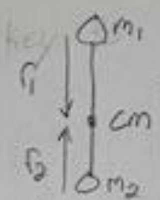
$$m_1 v = (m_1 + m_2) V_{cm}$$

$$\Rightarrow V_{cm} = \frac{m_1 v}{m_1 + m_2}$$

Conservation of angular momentum

- Use center of mass at time of collision as the reference point

Note: cm is not in the middle of the rod unless $m_1 = m_2$



$$\Rightarrow \quad r_1 = \frac{m_2}{m_1 + m_2} l$$

$$r_2 = \frac{m_1}{m_1 + m_2} l$$

→ Initial angular momentum

$$L_i = r \times p = m_1 v r_1 = \frac{m_1 m_2}{m_1 + m_2} v l$$

$$L_f = I_{dombell}\, \omega = (m_1 r_1^2 + m_2 r_2^2)\omega = \frac{m_1 m_2}{m_1 + m_2} l^2 \omega$$

$$L_i = L_f = \frac{m_1 m_2}{m_1 + m_2} v l = \frac{m_1 m_2}{m_1 + m_2} l^2 \omega \Rightarrow \frac{v}{l} = \omega$$

Now construct final KE

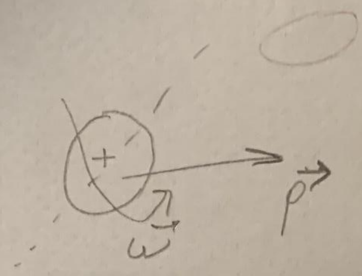
$$KE_f = KE_{trans} + KE_{rot} = \frac{1}{2}(m_1 + m_2) V_{cm}^2 + \frac{1}{2} I \omega^2 \Rightarrow$$

$$\frac{1}{2}(m_1 + m_2)\left(\frac{m_1 v}{m_1 + m_2}\right)^2 + \frac{1}{2}\left[\frac{m_1 m_2}{m_1 + m_2} l^2\right]\left(\frac{v}{l}\right)^2 \Rightarrow \boxed{\frac{1}{2} m_1 v^2 = KE_i}$$

Problem 13

Problem 4

7/27



$$\Delta E = 0 = \Delta KE_{trans} + \Delta KE_{rot} + \Delta U$$

OR

Initial

Final

$$KE_{rot,i} + KE_{trans,i} + U_i = U_f + KE_{trans,f} + KE_{rot}$$

$$-k\frac{q_1 q_2}{|\vec{r}_{source} - \underbrace{\vec{r}_1}_{=0}|} = \frac{-k q_1 q_2}{|\vec{r}_{source} - \vec{r}_2|} + \frac{1}{2}m|\vec{v}|^2 + \frac{1}{2}I\omega^2$$

$$\frac{\frac{k q_1 q_2}{|\vec{r}_{source}|} - \frac{k q_1 q_2}{|\vec{r}_{source} - \vec{r}_2|} - \frac{1}{2}\frac{|\vec{p}|^2}{M}}{\frac{1}{2}\left(\frac{2}{5}MR^2\right)} = \omega^2$$

$$\frac{k q_1 q_2 \left(\frac{1}{|\vec{r}_{source}|} - \frac{1}{|\vec{r}_{source} - \vec{r}_2|} \right) - \frac{1}{2}\frac{|\vec{P}|^2}{M}}{\frac{1}{5} M R^2}$$

$$\frac{\left[\left(9 \cdot 10^9 \frac{kg \cdot m^3}{s^2 \cdot C^2}\right)\left(2 \cdot 10^{-3}\ C\right)\left(-2 \cdot 10^{-3}\ C\right)\left(\frac{1}{\sqrt{1^2 + 1^2}\ m} - \frac{1}{\sqrt{(1-.25)^2 + (1-.5)^2}\ m} \right) - \frac{1}{2}\frac{(120^2 + 120^2)\left(\frac{kg \cdot m}{s}\right)^2}{1\ kg} \right]}{\frac{1}{5}(1\ kg)(.2\ m)^2}$$

$$\Longrightarrow \quad 10321.3 = \omega^2$$

$$\boxed{\omega = 101.6\ \frac{rad}{s}}$$